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Subject: Extra Credit for proof of infinity norm
Date: April 8, 2020 at 1:14 AM
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KW

Dear Professor Towell,

As you mentioned earlier in one lecture recording, proof of the infinity norm can count for extra credit. Here's my proof:

Let $\|x\|_\infty = \max_{i=1..n} |x_i|$

and $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$

We wish to show that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty .$$

We have that

$$\|x\|_p = \|x\|_\infty \frac{\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}}{\|x\|_\infty}$$

$$= \|x\|_\infty \left(\sum_{i=1}^{n} \frac{|x_i|^p}{\|x\|_\infty^p}\right)^{\frac{1}{p}}$$

$$= \|x\|_\infty \left(\sum_{i=1}^{n} \left(\frac{|x_i|}{\|x\|_\infty}\right)^p\right)^{\frac{1}{p}}$$

$$\leq \|x\|_\infty \cdot n^{\frac{1}{p}} \quad \text{since } \left(\frac{|x_i|}{\|x\|_\infty}\right)^p \leq 1 \text{ for every } i \text{ as } \|x\|_\infty = \max_i |x_i| \geq |x_i|$$

i.e. We have $\|x\|_p \leq \|x\|_\infty \cdot n^{\frac{1}{p}}$

$$\lim_{p\to\infty} \|x\|_p \leq \lim_{n\to p} \left(\|x\|_\infty \cdot n^{\frac{1}{p}}\right)$$

$$\lim_{p\to\infty} \|x\|_p \leq \|x\|_\infty \lim_{p\to\infty} n^{\frac{1}{p}}$$

$$\lim_{p\to\infty} \|x\|_p \leq \|x\|_\infty \cdot 1$$

$$\boxed{\lim_{p\to\infty} \|x\|_p \leq \|x\|_\infty} \quad (1)$$

On the other hand,

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \geq \left[\left(\max_i |x_i|\right)^p\right]^{\frac{1}{p}} = \max_i |x_i| = \|x\|_\infty$$

since $\left(\max_i |x_i|\right)^p$ is one term

in the sum.

i.e. $\|x\|_p \geq \|x\|_\infty$

$$\lim_{p\to\infty} \|x\|_p \geq \lim_{p\to\infty} \|x\|_\infty$$

$$\boxed{\lim_{p\to\infty} \|x\|_p \geq \|x\|_\infty} \quad ②$$

Combine ① and ②, we have $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$

Now replace x with $x-y$, we have

$$\left(\sum_{i=1}^{n} (|x_i - y_i|)^p\right)^{\frac{1}{p}} = \max_i |x_i - y_i|$$

Thank you,
Kejing