CMSC 380

Graph Traversals and Search

Graph Traversals

- Graphs can be traversed breadth-first, depth-first, or by path length

- We need to specifically guard against cycles
  - Mark each vertex as "closed" when we encounter it and do not consider closed vertices again

Queuing Function

- Used to maintain a ranked list of nodes that are candidates for expansion
- Substituting different queuing functions yields different traversals/searches:
  - FIFO Queue : breadth first traversal
  - LIFO Stack : depth first traversal
  - Priority Queue : Dijkstra’s algorithm / uniform cost

Bookkeeping Structures

- Typical node structure includes:
  - vertex ID
  - predecessor node
  - path length
  - cost of the path

- Problem includes:
  - graph
  - starting vertex
  - goalTest(Node n) – tests if node is a goal state (can be omitted for full graph traversals)

General Graph Search / Traversal

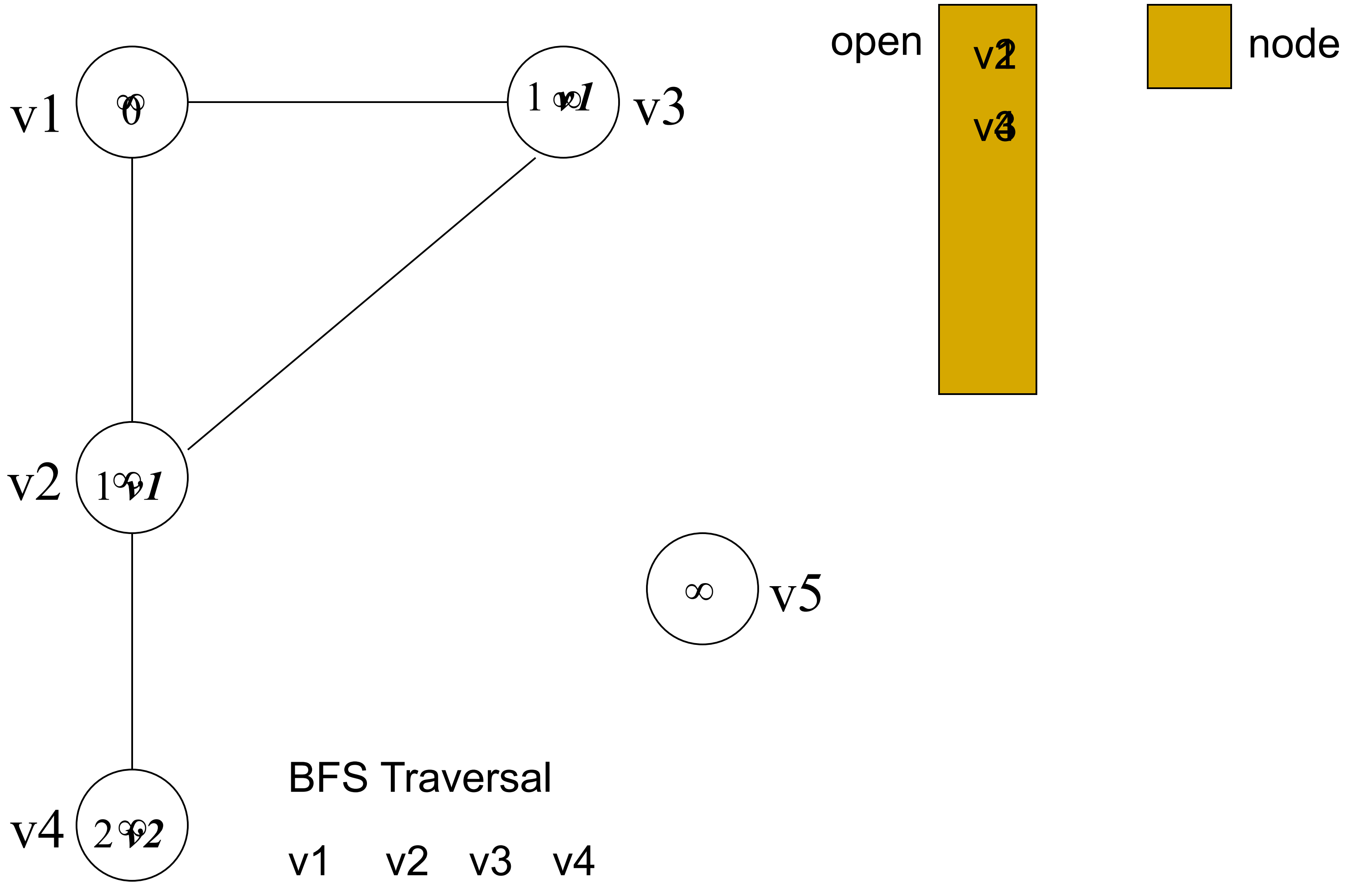
```
// problem describes the graph, start vertex, and goal test
// queueingfn is a comparator function that ranks two states
// graphSearch returns either a goal node or failure

graphSearch(problem, queuingFn) {
    open = {}, closed = {}

    queuingFn(open, new Node(problem.startvertex))  //initialize

    loop {
      if empty(open) then return FAILURE           //no nodes remain

      curr = removeFront(open)                     //get current node

      if problem.goalTest(curr.vertex)             //optional goaltest
         return curr                               //for search

      if curr.vertex is not in closed {            //avoid duplicates
         add curr.vertex to closed
         for each Vertex w adjacent to curr.vertex    // expand node
            queuingFn(open, new Node(w,curr));
      }
    }
}
```

Unweighted Shortest Path Problem

- Unweighted shortest-path problem: Given an unweighted graph G = (V, E) and a starting vertex s, find the shortest unweighted path from s to every other vertex in G.

- Breadth first search
  - Use FIFO queue
  - Finds shortest path if edges are unweighted (or equal cost)
  - Recover path by backtracking through nodes

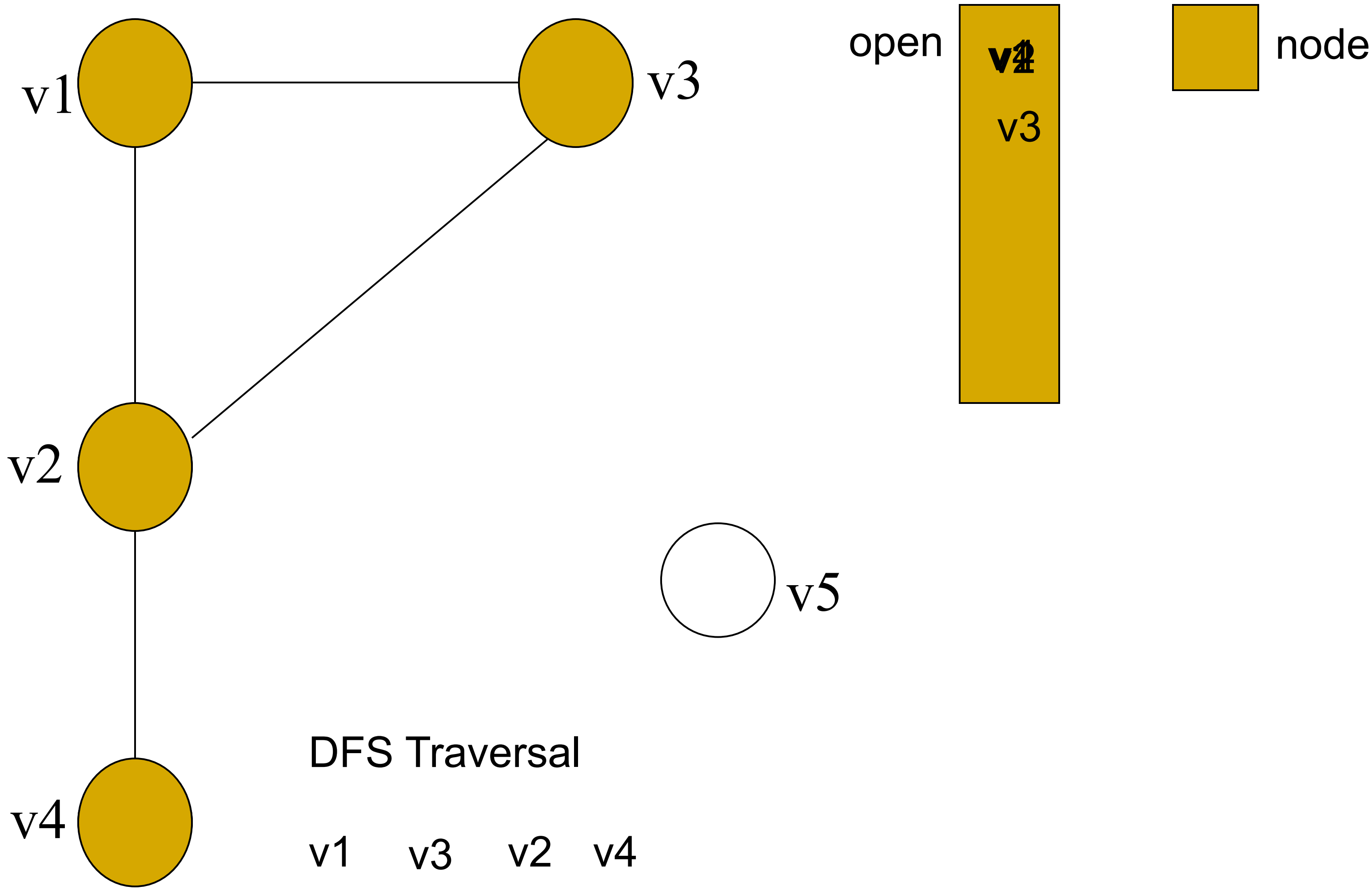
Breadth-First Example: Queue



BFS Traversal

v1 v2 v3 v4

DFS Example: Stack



Traversal Performance

- What is the performance of DF and BF traversal?
- Each vertex appears in the stack or queue exactly once in the worst case. Therefore, the traversals are at least O(|V|). However, at each vertex, we must find the adjacent vertices. Therefore, df- and bf-traversal performance depends on the performance of the `getAdjacent` operation.

GetAdjacent

- Method 1: Look at every vertex (except u), asking "are you adjacent to u?"

```
List<Vertex> L;
for each Vertex v except u
  if (v.isAdjacentTo(u))
    L.push_back(v);
```

- Assuming O(1) performance for `isAdjacentTo`, then `getAdjacent` has O(|V|) performance and traversal performance is O($|V^2|$)

GetAdjacent (2)

- Method 2: Look only at the edges which impinge on u. Therefore, at each vertex, the number of vertices to be looked at is deg(u), the degree of the vertex
- For this approach where `getAdjacent` is O(deg(u)).

  The traversal performance is

  $$O\left(\sum_{i=1}^{|V|} \deg(v_i)\right) = O(|E|)$$

  since `getAdjacent` is done O(|V|) times.
- However, in a disconnected graph, we must still look at every vertex, so the performance is O(|V| + |E|).

Weighted Shortest Path Problem

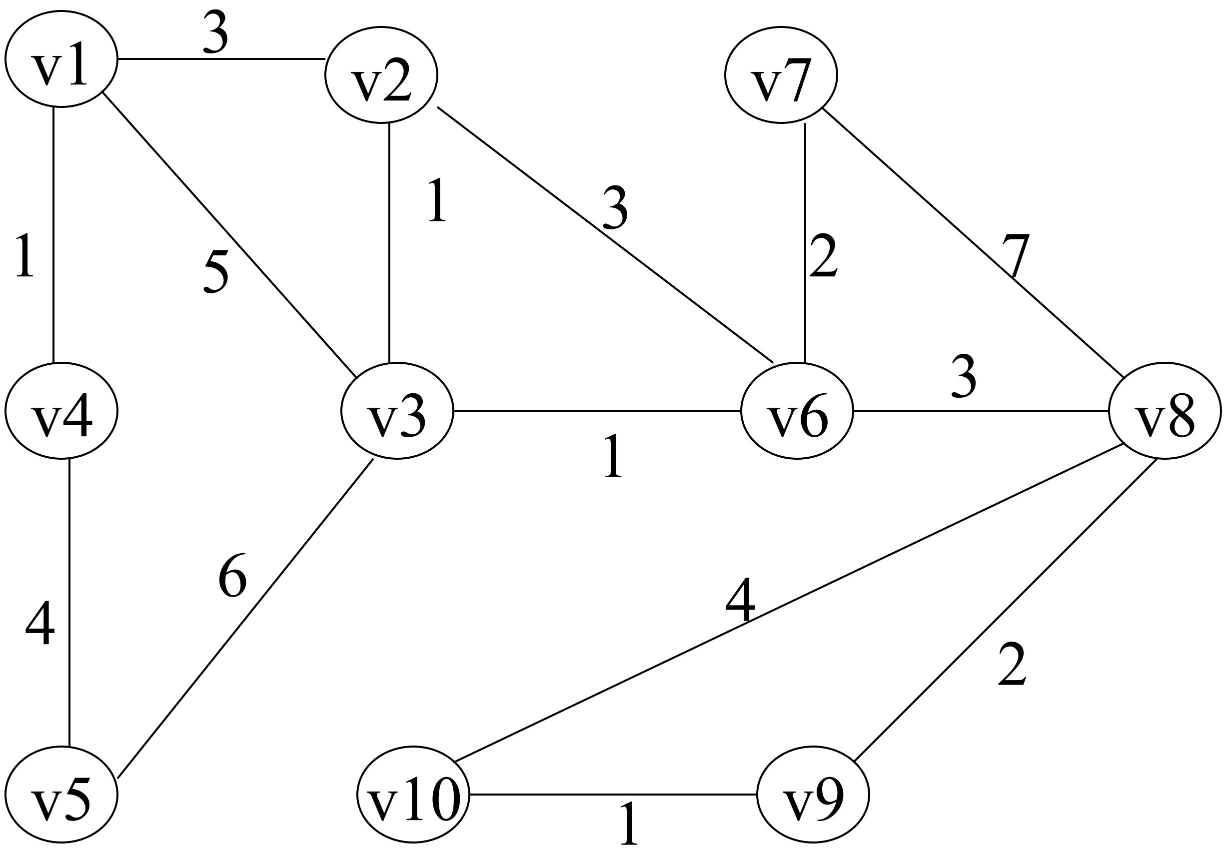
Single-source shortest-path problem:

Given as input a weighted graph, G = (V, E), and a distinguished starting vertex, s, find the shortest weighted path from s to every other vertex in G.

Dijkstra's algorithm (also called uniform cost search)

- Use a priority queue in general search/traversal
- Keep tentative distance for each vertex giving shortest path length using vertices visited so far.
- Record vertex visited before this vertex (to allow printing of path).
- At each step choose the vertex with smallest distance among the unvisited vertices (greedy algorithm).

Example Network



Dijkstra’s Algorithm

- The pseudo code for Dijkstra’s algorithm assumes the following structure for a Vertex object

```
class Vertex
{
  public List adj;       //Adjacency list
  public boolean known;
  public DisType dist;  //DistType is probably int
  public Vertex path;
  //Other fields and methods as needed
}
```

Dijkstra's Algorithm

```
void dijksra(Vertex start)
{
   for each Vertex v in V {
        v.dist = Integer.MAX_VALUE;
        v.known = false;
        v.path = null;
   }

   start.distance = 0;

   while there are unknown vertices {
        v = unknown vertex with smallest distance
        v.known = true;
        for each Vertex w adjacent to v
               if (!w.known)
                      if (v.dist + weight(v, w)< w.distance){
                             decrease(w.dist to v.dist+weight(v, w))
                             w.path = v;
                      }
   }
}
```

Correctness of Dijkstra's Algorithm

- The algorithm is correct because of a property of shortest paths:
- If $P_k = v_1, v_2, ..., v_j, v_k$, is a shortest path from v_1 to v_k, then $P_j = v_1, v_2, ..., v_j$, must be a shortest path from v_1 to v_j. Otherwise P_k would not be as short as possible since P_k extends P_j by just one edge (from v_j to v_k)
- P_j must be shorter than P_k (assuming that all edges have positive weights). So the algorithm must have found P_j on an earlier iteration than when it found P_k.
- i.e. Shortest paths can be found by extending earlier known shortest paths by single edges, which is what the algorithm does.

Running Time of Dijkstra's Algorithm

- The running time depends on how the vertices are manipulated.
- The main 'while' loop runs O(|V|) time (once per vertex)
- Finding the "unknown vertex with smallest distance" (inside the while loop) can be a simple linear scan of the vertices and so is also O(|V|). With this method the total running time is O ($|V|^2$). This is acceptable (and perhaps optimal) if the graph is dense (|E| = O ($|V|^2$)) since it runs in linear time on the number of edges.
- If the graph is sparse, (|E| = O (|V|)), we can use a priority queue to select the unknown vertex with smallest distance, using the deleteMin operation (O(lg |V|)). We must also decrease the path lengths of some unknown vertices, which is also O(lg|V|). The deleteMin operation is performed for every vertex, and the "decrease path length" is performed for every edge, so the running time is O(|E| lg|V| + |V|lg|V|) = O((|V|+|E|) lg|V|) = O(|E| lg|V|) if all vertices are reachable from the starting vertex

Dijkstra and Negative Edges

- Note in the previous discussion, we made the assumption that all edges have positive weight. If any edge has a negative weight, then Dijkstra's algorithm fails. Why is this so?
- Suppose a vertex, u, is marked as "known". This means that the shortest path from the starting vertex, s, to u has been found.
- However, it's possible that there is negatively weighted edge from an unknown vertex, v, back to u. In that case, taking the path from s to v to u is actually shorter than the path from s to u without going through v.
- Other algorithms exist that handle edges with negative weights for weighted shortest-path problem.

Directed Acyclic Graphs

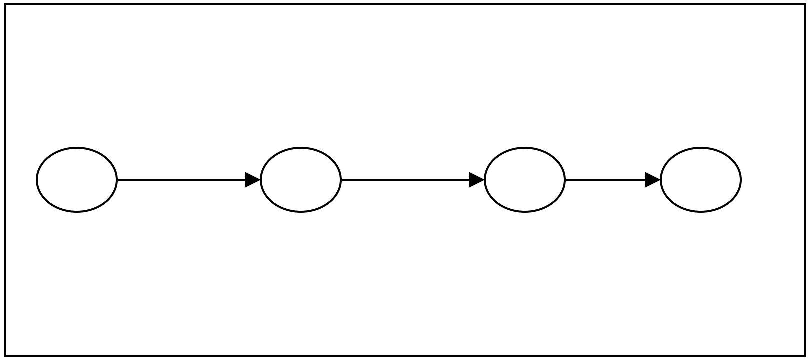
- A ***directed acyclic graph*** is a directed graph with no cycles.
- A ***strict partial order*** R on a set S is a binary relation such that
  - for all a∈S, aRa is false (irreflexive property)
  - for all a,b,c ∈S, if aRb and bRc then aRc is true (transitive property)
- To represent a partial order with a DAG:
  - represent each member of S as a vertex
  - for each pair of vertices (a,b), insert an edge from a to b if and only if a R b

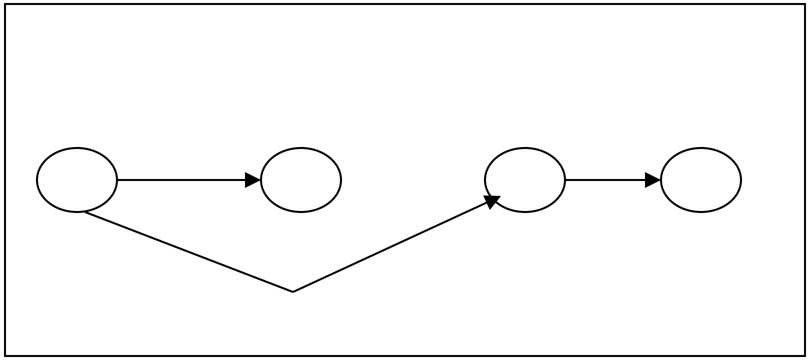
More Definitions

- Vertex i is a ***predecessor*** of vertex j if and only if there is a path from i to j.
- Vertex i is an ***immediate predecessor*** of vertex j if and only if (i, j) is an edge in the graph.
- Vertex j is a ***successor*** of vertex i if and only if there is a path from i to j.
- Vertex j is an ***immediate successor*** of vertex i if and only if (i, j) is an edge in the graph.

Topological Ordering

- A topological ordering of the vertices of a DAG G = (V,E) is a linear ordering such that, for vertices i, j ∈V, if i is a predecessor of j, then i precedes j in the linear order,
i.e. if there is a path from v_i to v_j, then v_i comes before v_j in the linear order

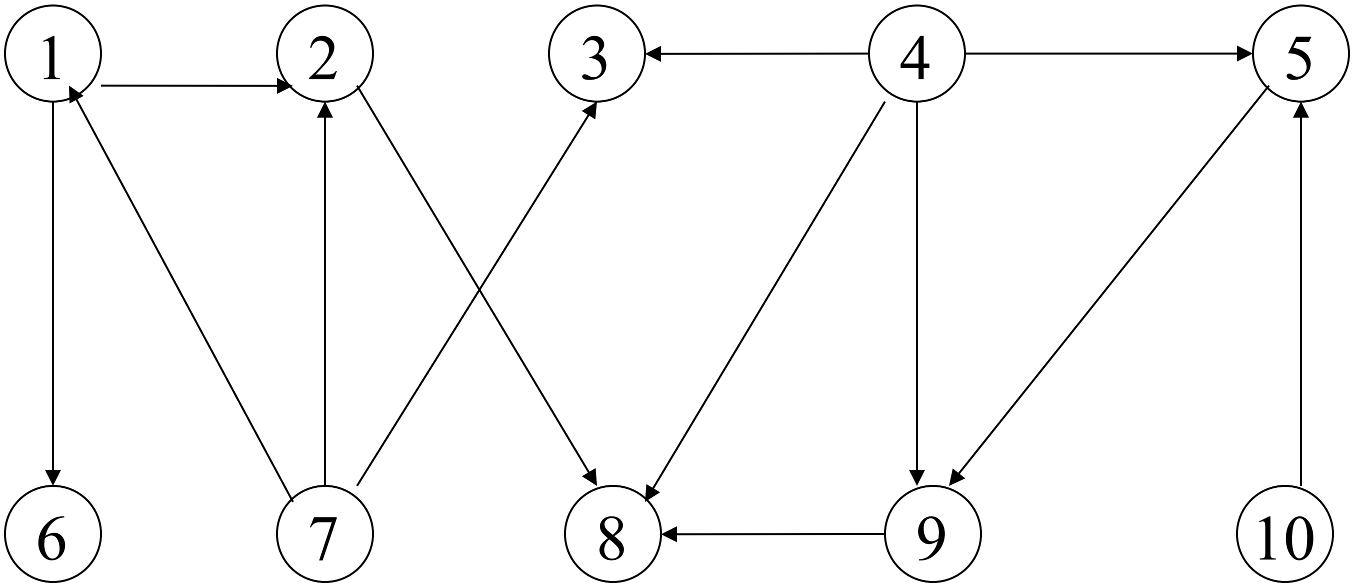




Topological Sort

```
void topsort( ) throws CycleFoundException
{
    Queue<Vertex> q = new Queue<Vertex>( );
    int counter = 0;

    for each Vertex v
        if( v.indegree == 0 )
            q.enqueue( v );

    while( !q.isEmpty( ) )
    {
        Vertex v = q.dequeue( );
        v.topNum = ++counter;  // Assign next number

        for each Vertex w adjacent to v
            if( --w.indegree == 0 )
                q.enqueue( w );
    }

    if( counter != NUM_VERTICES )
        throw new CycleFoundException( );
}
```

TopSort Example



Running Time of TopSort

1. At most, each vertex is enqueued just once, so there are $O(|V|)$ constant time queue operations.
2. The body of the for loop is executed at most once per edges = $O(|E|)$
3. The initialization is proportional to the size of the graph if adjacency lists are used = $O(|E| + |V|)$
4. The total running time is therefore $O(|E| + |V|)$