# Computational Geometry

Gauss-Bonnet

Cauchy Rigidity

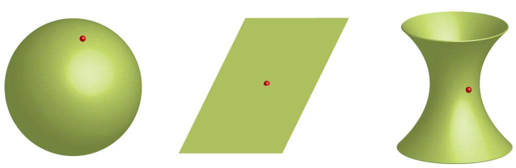
## Gauss-Bonnet

- Let $S$ be a smooth surface without boundary.

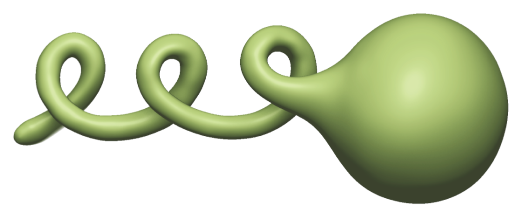
$$\int_S K\, dA = 2\pi\, \chi(S)$$

- $K$ is the Gaussian curvature
- $dA$ is the element of area

## Curvature

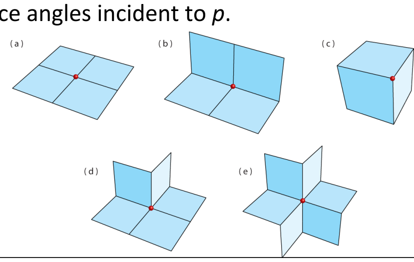


## Total Curvature = 4π



## Polyhedral Gaussian Cuvature

- The Gaussian curvature $K(p)$ at a point $p$ of a polyhedral surface is 2π minus the sum of the face angles incident to $p$.




## Polyhedral Gauss-Bonnet

- For a polyhedron $P$,

$$\Sigma_{v\in P}\, K(v) = 2\pi\, \chi(P)$$

## Extension to Surface without Boundary

- Let $S$ be a smooth surface.

$$\int_S K\, dA + \int_{\delta s} K_g\, ds = 2\pi\, \chi(S)$$

- $K_g$ is the geodesic curvature on a curve C
- $\delta s$ is the boundary curve

## Geodesic Curvature

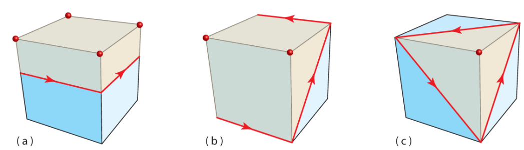
- Let $P$ be a polyhedron with boundary. The geodesic curvature $K(p)$ at a boundary point $p$ of $P$ is $\pi$ minus the sum of the face angles incident to $p$.
- Measures the turn angle at each $p$ on a discrete boundary curve.
- Solely concentrated on the corners of the boundary curve.

## General Polyhedral Gauss-Bonnet

- For a polyhedron $P$ with $\delta P$ as its boundary and $P \setminus \delta P$ its interior.

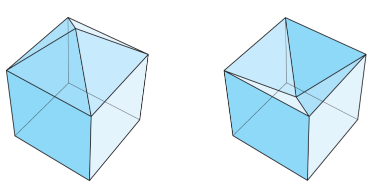
$$\Sigma_{v \in P \setminus \delta P} K(v) + \Sigma_{v \in \delta P} K(v) = 2\pi\, \chi(P)$$

## Three Examples




## Polyhedral Congruency

- congruent faces
- similiary arranged about each vertex
- convexity



## Cauchy Rigidity Theorem

- If two closed, convex polyhedra are combinatorially equivalent, with corresponding faces congruent, then the polyhedra are congruent. In particular, the dihedral angles at corresponding edges are the same.
- Gluing together a collection of flat, rigid polygonal faces so that every vertex has non-negative Gaussian curvature will result in a unique convex polyhedron.

## Steffen Polyhedron