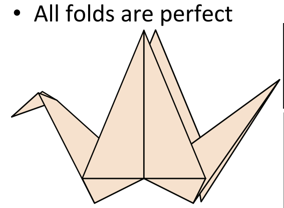
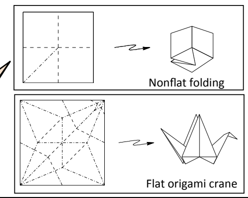
Computational Geometry

Mathematics of Origami

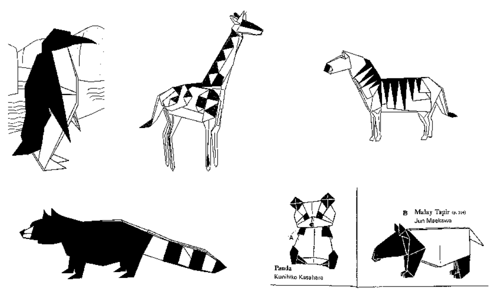
Flat Origami

- Can be pressed in a book without new creases
- Paper has 0 thickness and creases have 0 width
- All folds are perfect

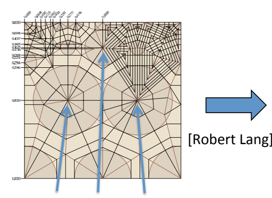




Flat Foldings of Single Sheets of Paper



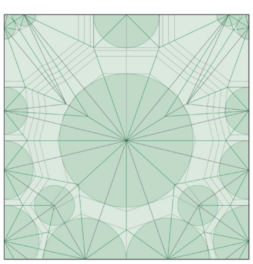
Moose



[Robert Lang]

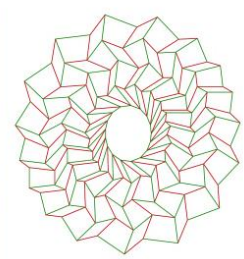
Flat vertex folds

Scorpion



Tessellation

(a)

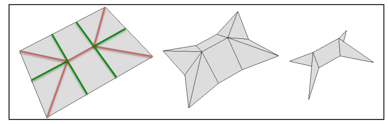


(b)

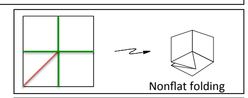
Plants

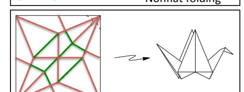


Structure of Foldings

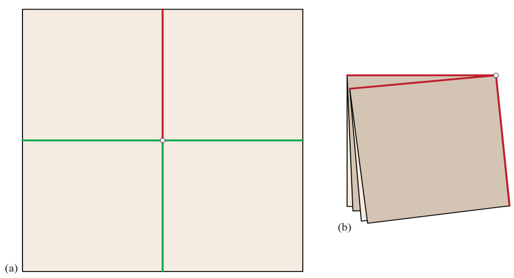


- Crease pattern: Mountain-Valley assignment

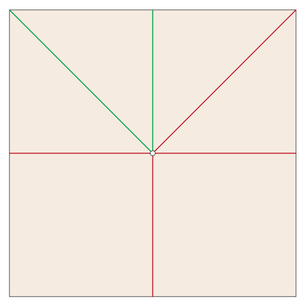




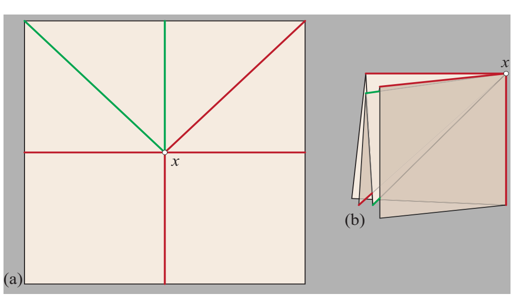
Degree 4



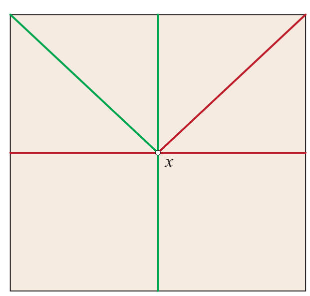
Degree 6 Vertex



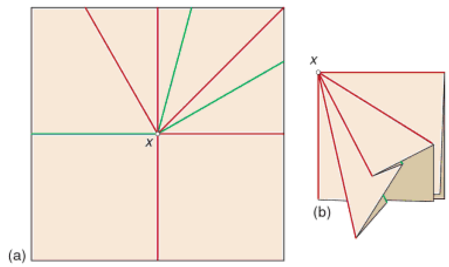
Flat Folding



Another Degree 6 Vertex

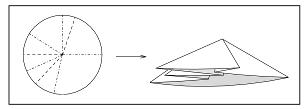


Degree 8



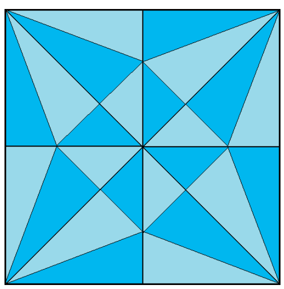
Single-Vertex Origami

- Consider a disk surrounding a lone vertex in a crease pattern
- When can it be folded flat?



Even Degree

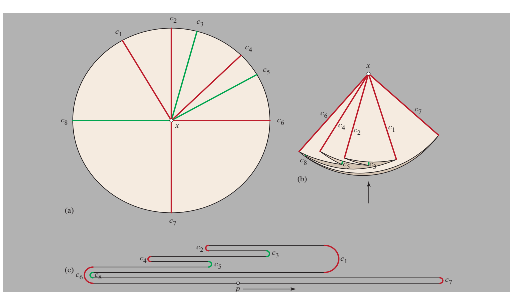
- A flat-folding vertex has even degree
- Crease patterns are two colorable



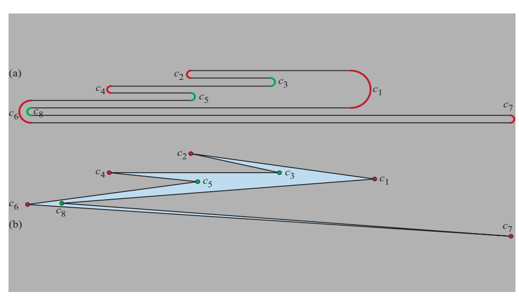
Maekawa-Justin

- If M mountain creases and V valley creases surround a vertex, then vertex is flat foldable if |M-V| = 2.

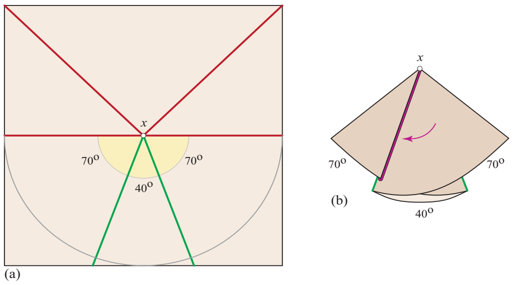
Maekawa-Justin



Maekawa-Justin



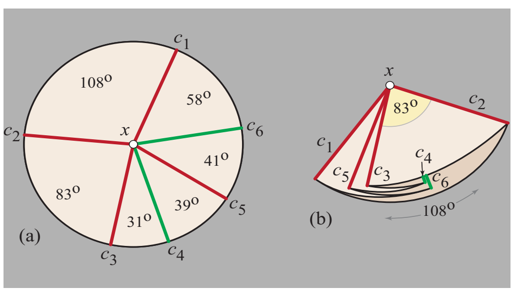
Flat Folding?



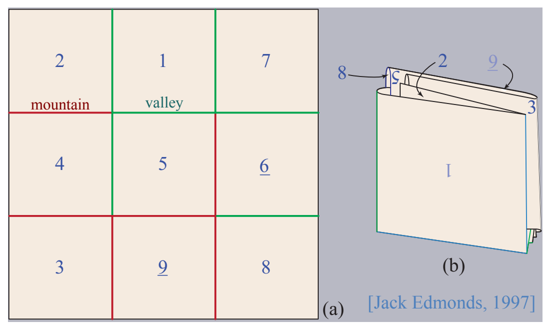
Kawasaki-Justin

- A vertex is flat-foldable if sum of alternate angles is 180°
  ($\Theta_1 + \Theta_3 + \ldots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \ldots + \Theta_n$)
  - Tracing disk's boundary along folded arc moves $\Theta_1 - \Theta_2 + \Theta_3 - \Theta_4 + \ldots + \Theta_{n-1} - \Theta_n$
  - Should return to starting point ⇒ equals 0
- If one angle is smaller than its two neighbors, the two surrounding creases must have opposite direction
  - Otherwise, the two large angles would collide

Kawasaki-Justin Theorem



Map Folding

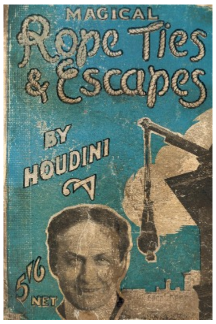


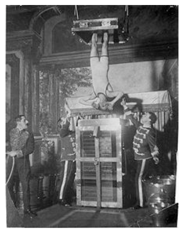
Map Folding

- What is the computational complexity of deciding if a given mountain-valley assignment on a regular grid of creases has a flat-folded state?
- Unsolved even on $n \times 2$ grids!
- NP-complete for folding orthogonal polygons with orthogonal creases
  - [Arkin, Demaine, Demaine, Mitchell, Sethia, Skiena, 2004]

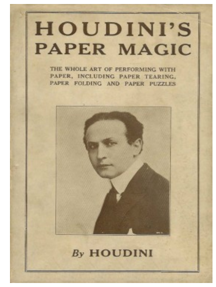
The Fold & One-Cut Theorem

Harry Houdini (1874-1926)





1922



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THE FIVE-POINTED STAR

To make a five-pointed star with one stroke of the scissors or a single tear, take a square of paper about one-half larger than the size of star desired and fold it in half as shown by *A* in Fig. 73, the folded edge being at the bot-

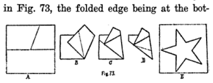

Fig 73

tom. Now fold the right-hand end toward the left on a line running from the centre at the bottom to a point one-third of the distance from the right-hand corner at the top, as shown by the dotted line on *A*, which will give you *B*. Then fold from right to left on the dotted line shown in *B*, thus forming *C*. After this, fold the left-hand corner toward the right *underneath* the other folds and the result will be as in *D*. By cutting or tearing along the

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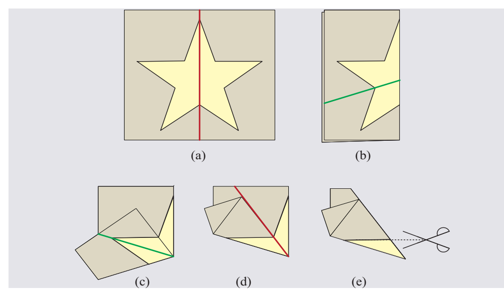
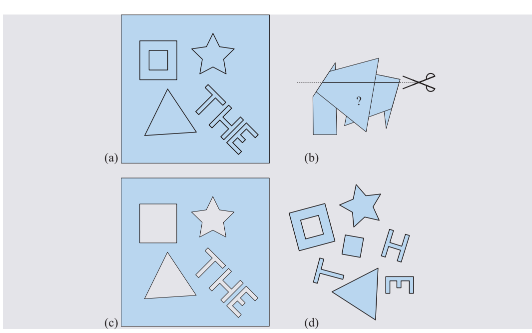
The Five-Pointed Star 177

dotted line there shown and unfolding the three-cornered section thus obtained, a perfect five-pointed star will be the result, as shown in *E*.

Care must be exercised in making the folds on exact lines, otherwise the points will be of unequal length and size. A little practice will enable you to make the folds and tear out a perfect star in from ten to twenty seconds.

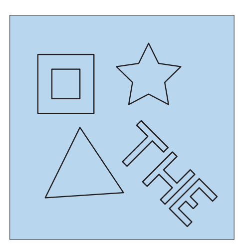


Fold & One-Cut Theorem

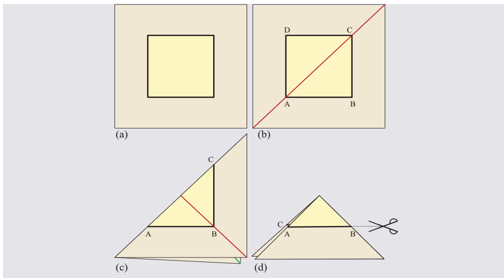


Theorem [Demaine, Demaine, Lubiw 1998]
[Bern, Demaine, Eppstein, Hayes 1999]
[Bern, Demaine, Eppstein, Hayes, O'R 2006]

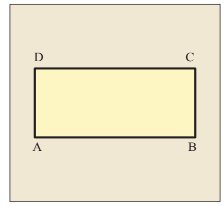
- Every straight-line plane graph can be folded flat so that one straight cut through, cuts out exactly the edges and vertices of the graph, and nothing more.



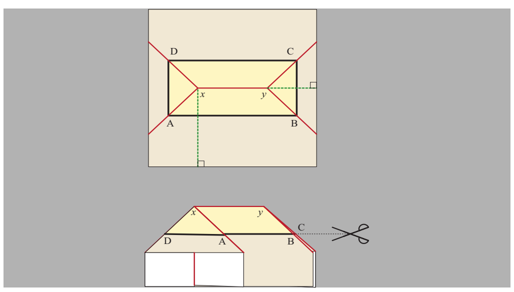
Square



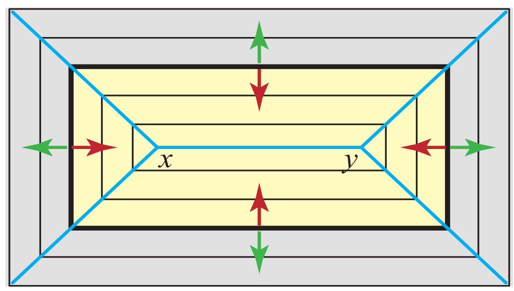
Rectangle



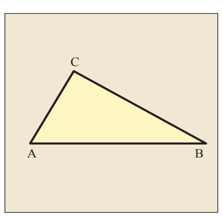
Rectangle

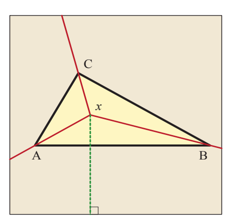


Rectangle Skeleton

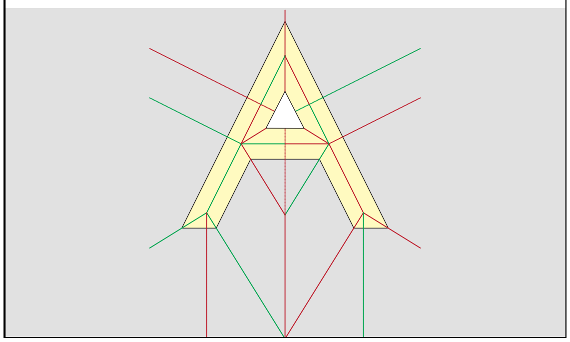


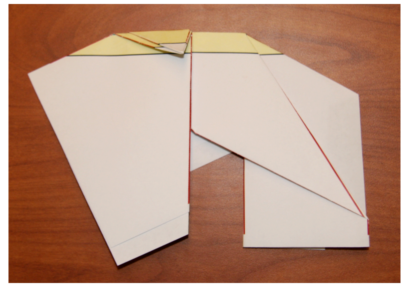
Irregular Triangle

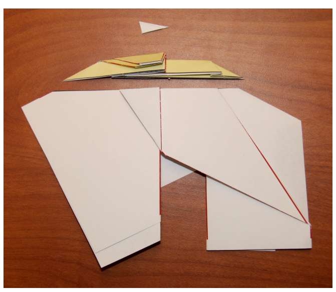


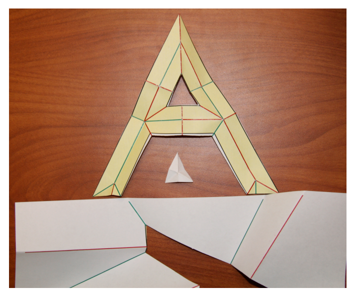


Letter ‘A’

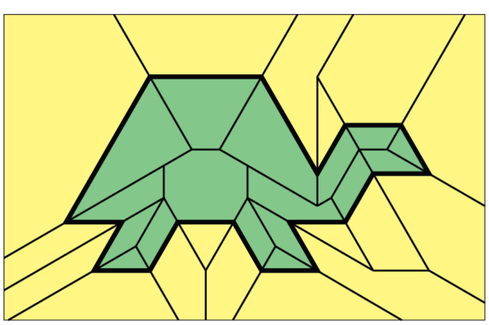




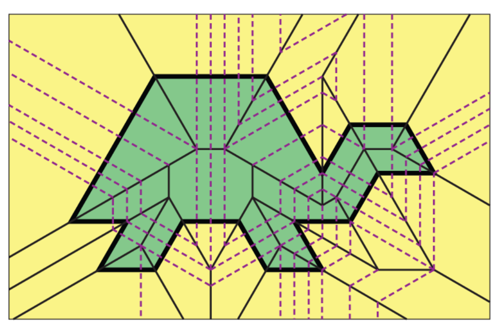


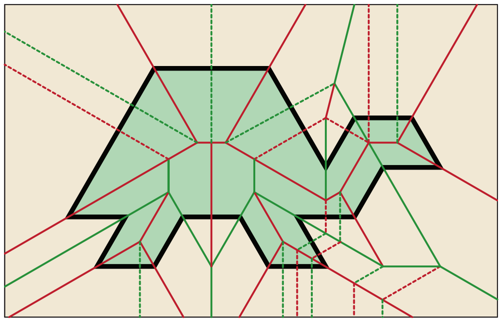


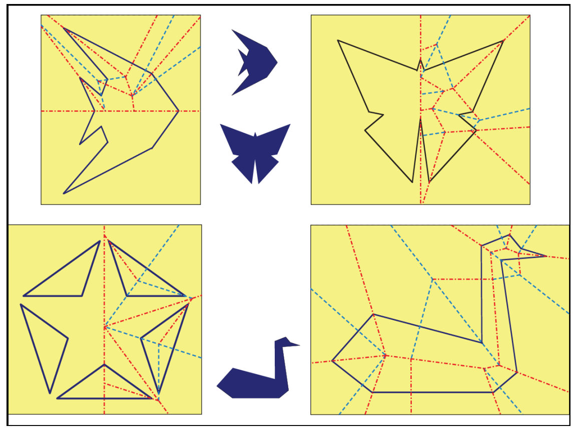
Straight Skeleton



Perpendiculars

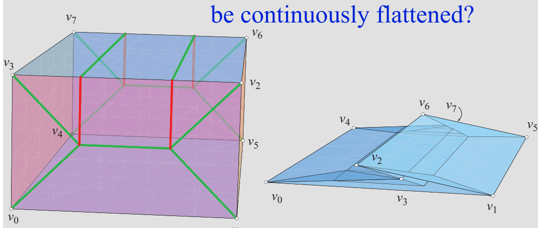






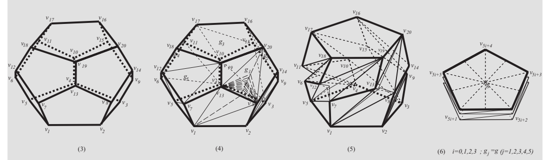
Collapsing polyhedron to flat folded state

Question: Can every polyhedron be continuously flattened?



(a) (b)

Theorem:
Convex polyhedra can be continuously flattened.



[Itoh, Nara, Vilcu, 2011]

Theorem:
Any polyhedral 2-manifold can be realized as flat origami. [Bern, Hayes, 2011]

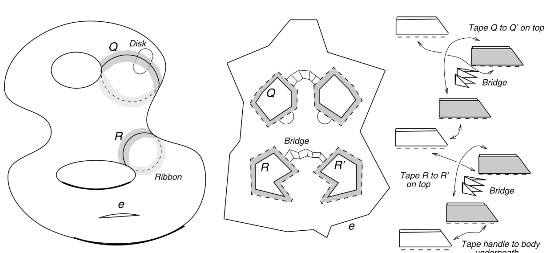


Fig. 8