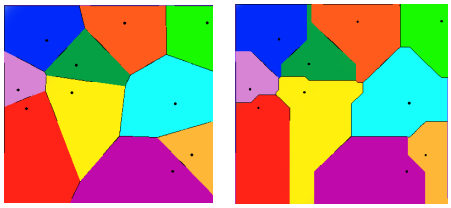
Computational Geometry

Voronoi Diagrams, Delaunay Triangulations, and Convex Hull

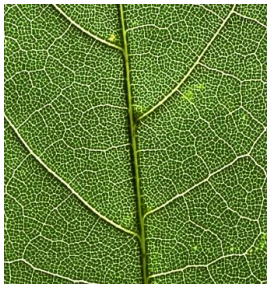
Convexity of the Voronoi Regions

- Heavily dependent on distance metric

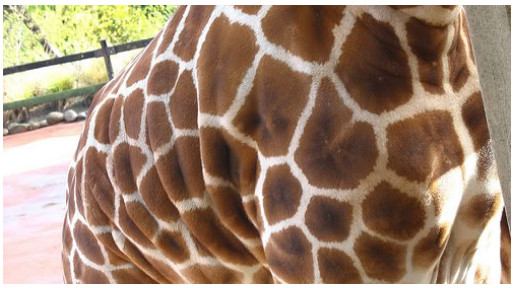


Spider Web

Leaf



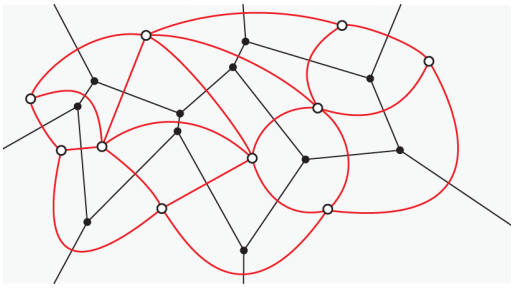
Giraffe



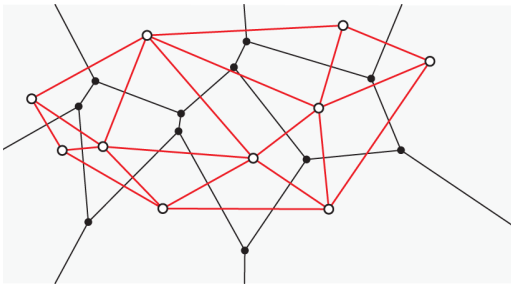
3D

- Nature by Numbers
- Animated 3D Voronoi

The Dual Graph

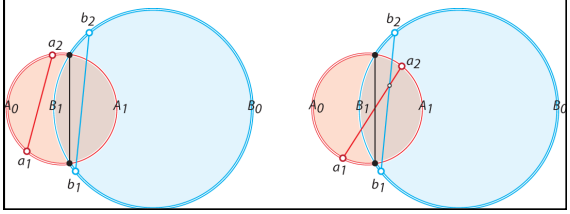


Straight-line Dual Graph



Lemma

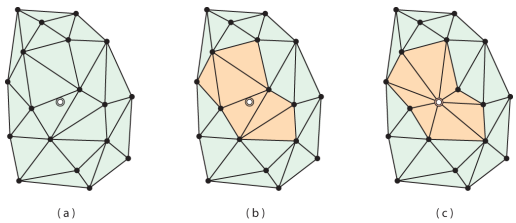
- Let A and B be two circles with chords that properly cross. Then at least one endpoints of one circle's chord is strictly inside the other circle.



Theorem

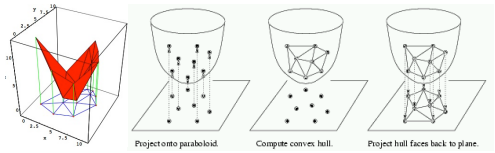
- The straight-line dual graph of *Vor*(*S*) is planar
- The straight-line dual graph of *Vor*(*S*) is a triangulation of *S* when *S* is in general position.
- The dual triangulation of *Vor*(*S*) is the Delaunay triangulation of *S.*

Incremental Delaunay via the Dual Graph



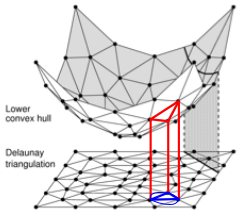
Delaunay and Convex Hull

- Lift sites to a paraboloid ($z = x^2 + y^2$)
- Compute 3D convex hull of points
- Project lower hull faces back to plane



Theorem

- Given a point set *S* in the plane, the Delaunay triangulation Del(S) is exactly the projection to the *xy*-plane of the lower convex hull of the points (x, y, x^2+y^2).
- applet

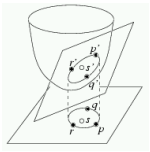


Proof

- Tangent plane to the paraboloid at (a, b, a^2+b^2)
  - take derivatives of $z = x^2 + y^2$
    - $\delta z/\delta x = 2x \Rightarrow (2a)$
    - $\delta z/\delta y = 2y \Rightarrow (2b)$
  - plane equation : $z = 2ax + 2by - (a^2+b^2)$
- Shift plane upwards in Z+ by r^2
  - $z = 2ax + 2by - (a^2+b^2) + r^2$
- Find intersection with the paraboloid

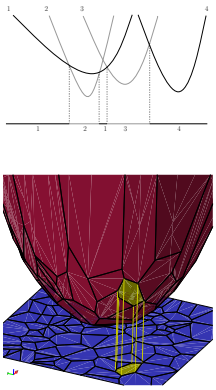
Proof

- A lower face *f* on the convex hull projects to a triangle with a circumcircle of radius *r*. Since *f* is on the lower hull, all other sites lie above (>*r* w.r.t. plane of tangency), and thus project outside of the circle, which then satisfies the empty circle property.

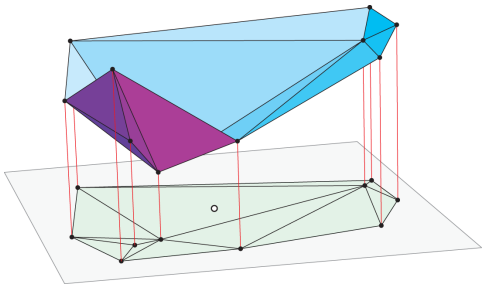


Notes

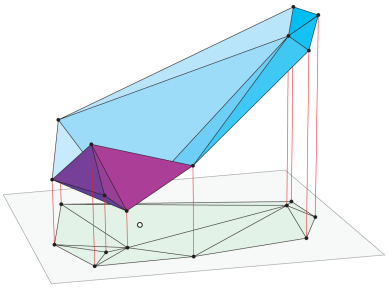
- If the tangent planes are also constructed and their intersections projected, it's the Voronoi diagram



Original



Translation to the Right



Notes

- Compute Delaunay by computing 3D convex hull instead – $O(n \log n)$
- The relationship holds in higher dimensions as well, thus Delaunay tetrahedralizations are typically constructed by constructing 4D convex hulls.