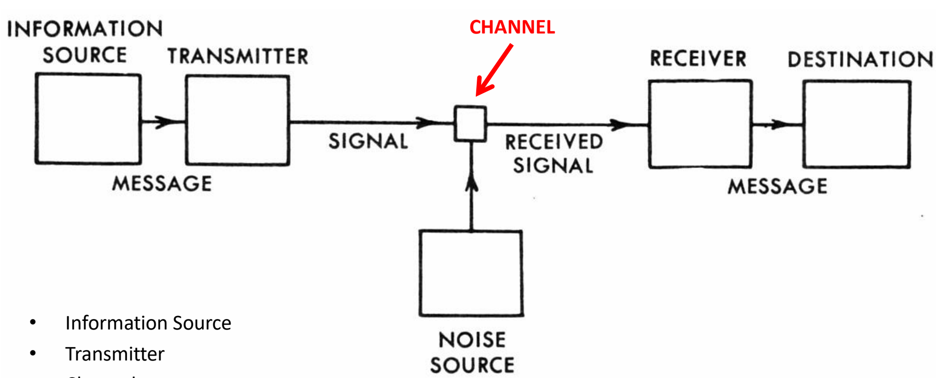
Introduction to Information Theory

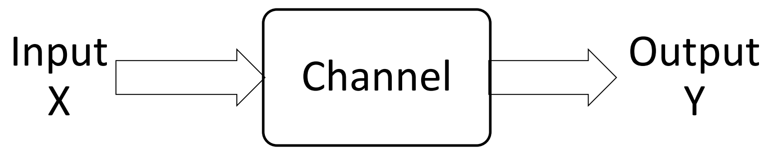
Part 4

A General Communication System

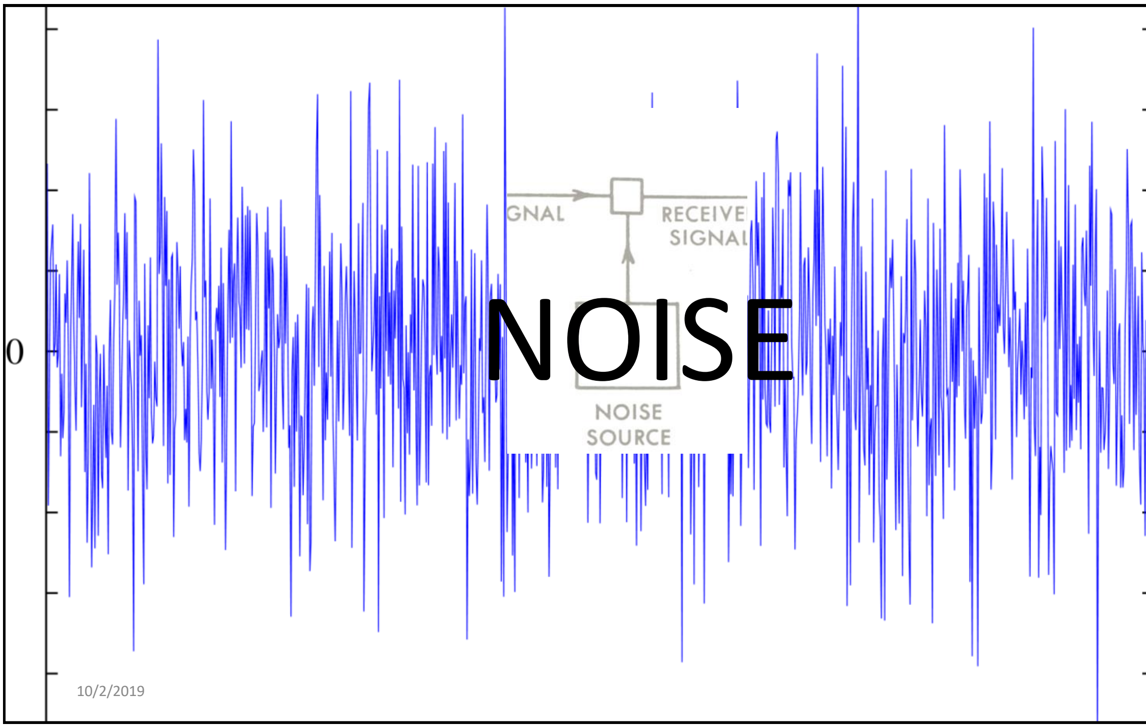


- Information Source
- Transmitter
- Channel
- Receiver
- Destination

Information Channel



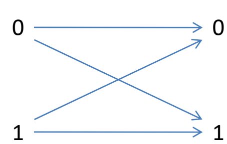
Perfect Communication (*Discrete Noiseless Channel*)



NOISE

Motivating Noise…

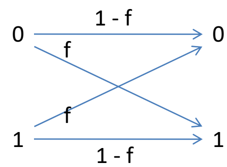
X
Transmitted
Symbol



Y
Received
Symbol

Motivating Noise…

f = 0.1, n = ~10,000



Motivating Noise…

Message: $5213.75
Received: $5293.75

1. Detect that an error has occurred.

2. Correct the error.

3. Watch out for the overhead.

Error Detection by Repetition

In the presence of 20% noise…

Message : $ 5 2 1 3 . 7 5
Transmission 1: $ 5 2 9 3 . 7 5
Transmission 2: $ 5 2 1 3 . 7 5
Transmission 3: $ 5 2 1 3 . 1 1
Transmission 4: $ 5 4 4 3 . 7 5
Transmission 5: $ 7 2 1 8 . 7 5

There is no way of knowing where the errors are.

Error Detection by Repetition

In the presence of 20% noise…

Message : $ 5 2 1 3 . 7 5
Transmission 1: $ 5 2 9 3 . 7 5
Transmission 2: $ 5 2 1 3 . 7 5
Transmission 3: $ 5 2 1 3 . 1 1
Transmission 4: $ 5 4 4 3 . 7 5
Transmission 5: $ 7 2 1 8 . 7 5
Most common: $ 5 2 1 3 . 7 5

1. Guesswork is involved.
2. There is overhead.

Error Detection by Repetition

In the presence of 50% noise…

Message : $ 5 2 1 3 . 7 5

…

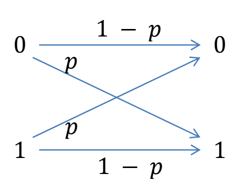
Repeat 1000 times!

1. Guesswork is involved.
   But it will almost never be wrong!

2. There is overhead.
   A LOT of it!

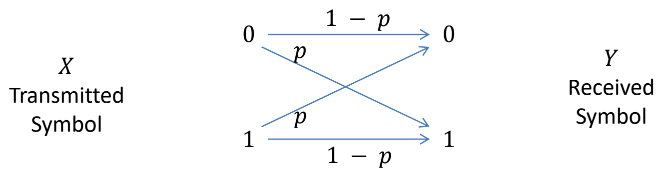
Binary Symmetric Channel (BSC) (Discrete Memoryless Channel)

X
Transmitted Symbol



Y
Received Symbol

Binary Symmetric Channel (BSC)
(Discrete Memoryless Channel)

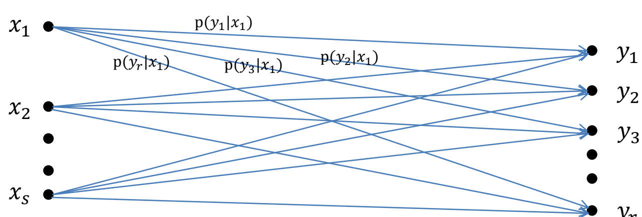


Defined by a set of **conditional probabilities** (*aka* **transitional probabilities**)

$$p(y|x) \, for \, all \, x \in X \, and \, y \in Y$$

The probability of y occurring at the output when x is the input to the channel.

A General Discrete Channel

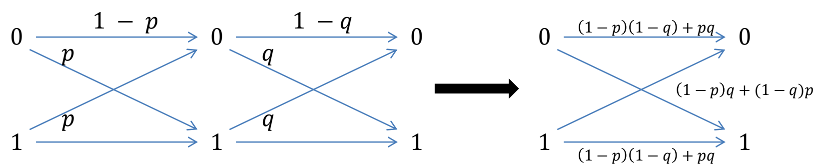


$s \times r$ transition probabilities

s input symbols

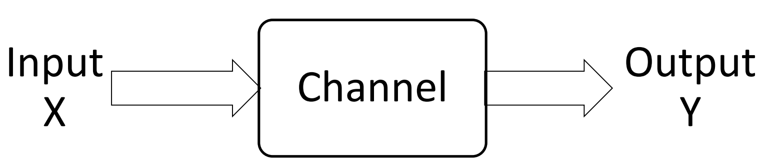
r output symbols

Channel With Internal Structure



Channel

- A channel can be modeled using a probabilistic model of source and what was received.



Conditional & Joint Entropy

- X ,Y random variables with entropy $H(X)$ and $H(Y)$
- **Conditional Entropy:** Average entropy in Y, given knowledge of X.

$$\boldsymbol{H(Y|X) = \sum_{x_i \in X} \sum_{y_j \in Y} p(x_i, y_j) \, log \frac{1}{p(y_{j|x_i})}}$$

where $p(x_i, y_j) = p(y_j | x_i) p(x_i)$

- **Joint Entropy:** $\boldsymbol{H(X, Y) = H(Y|X) + H(X)}$
  Entropy of the pair (X, Y)

Mutual Information

- The mutual information of a random variable X given the random variable Y is

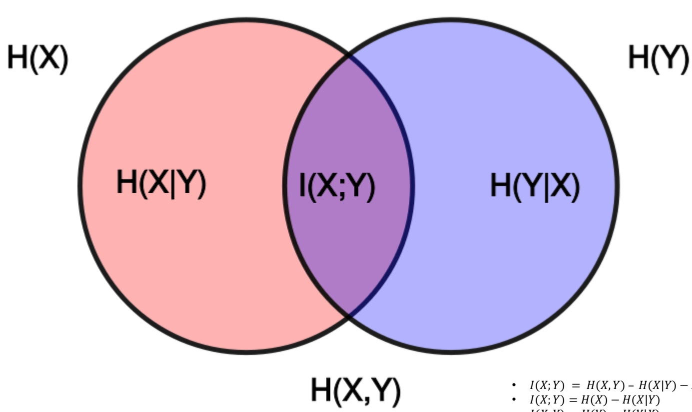
$$I(X; Y) = H(X) - H(X|Y)$$

It is the information about X transmitted by Y.

Mutual Information: Properties

- $I(X;Y) \; = \; H(X,Y) - H(X|Y) - H(Y|X)$
- $I(X;Y) = H(X) - H(X|Y)$
- $I(X;Y) = H(Y) - H(Y|X)$
- $I(X;Y)$ is symmetric in X and Y
- $I(X;Y) = \sum_x \sum_y p(x,y) log \frac{p(x,y)}{p(x)p(y)}$
- $I(X;Y) >= 0$
- $I(X;X) \; = \; H(X)$

Entropy Concepts



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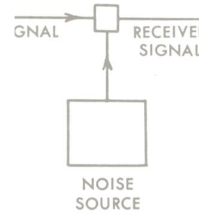
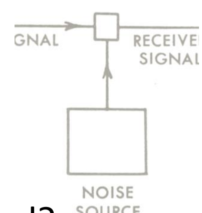
Channel Capacity



- What is the capacity of the channel?
- What is the reliability of communication across the channel?

Channel Capacity



- What is the capacity of the channel?
- What is the reliability of communication across the channel?
- Defined **Channel Capacity** as the rate at which reliable communication is possible.
- Capacity is defined in terms of **Mutual Information** I(X, Y)
- Mutual Information is defined in terms of entropy of source H(X) and the **joint entropy** H(X, Y)
- Joint entropy is defined in terms of source entropy H(X) and the **conditional entropy** H(Y|X)

Channel Capacity

- The capacity of a channel is the maximum possible mutual information that can be achieved between input and output by varying the probabilities of the input symbols.

  If X is the input channel and Y is the output, the capacity C is

$$\boldsymbol{c} = \max_{\boldsymbol{input\ probabilities}} \boldsymbol{I(X;Y)}$$

Channel Capacity

$$\boldsymbol{c} = \max_{\boldsymbol{input\ probabilities}} \boldsymbol{I(X;Y)}$$

Mutual information about X given Y is the information transmitted by the channel and depends on the probability structure

– Input probabilities
– Transition probabilities
– Output probabilities

Channel Capacity

$$c = \max_{input\ probabilities} I(X;Y)$$

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– Transition probabilities: fixed by properties of channel
– Output probabilities: determined by input and transition probabilities

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- Input probabilities: can be adjusted by suitable coding
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- Output probabilities: determined by input and transition probabilities

That is, input probabilities determine mutual information and can be varied by coding. The maximum mutual information with respect to these input probabilities is the channel capacity.

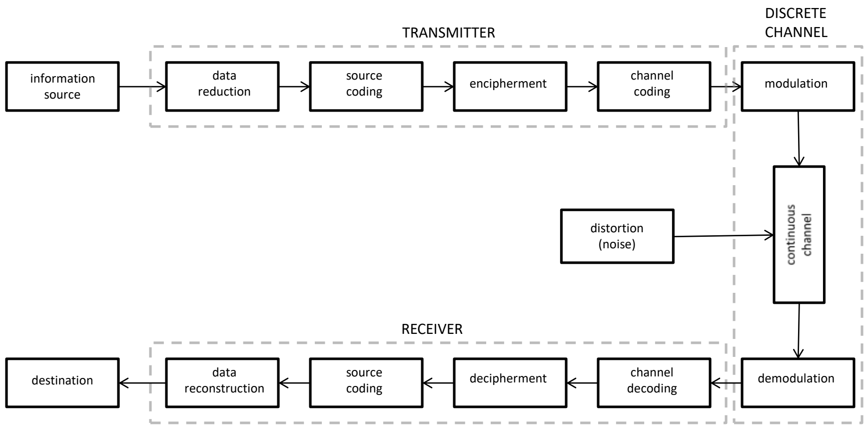
Shannon's Second Theorem

- Suppose a discrete channel has capacity C and the source has entropy H

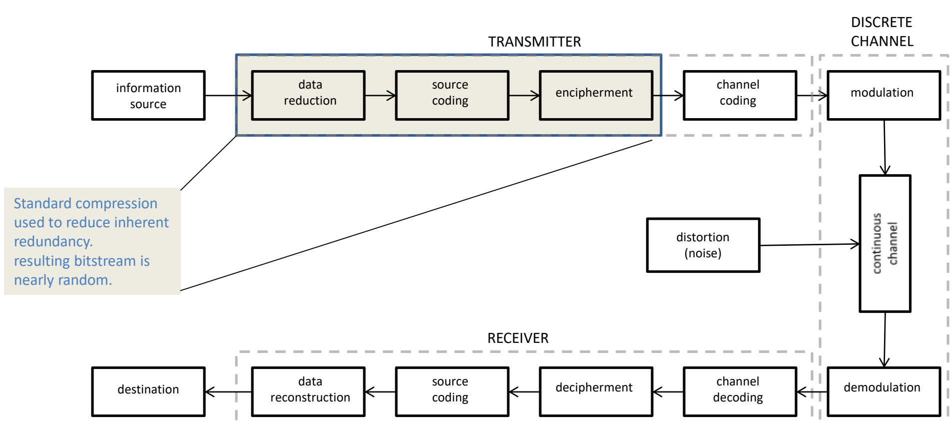
  If H < C there is a coding scheme such that the source can be transmitted over the channel with an arbitrarily small frequency of error.

  If H > C, it is not possible to achieve arbitrarily small error frequency.

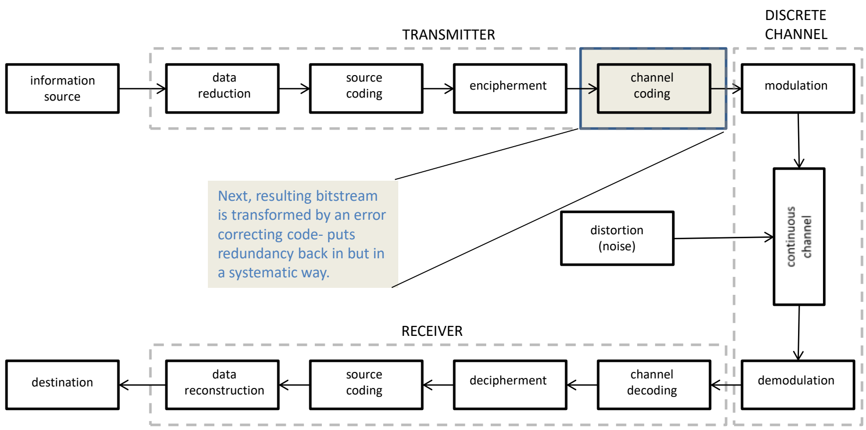
Detailed Communication Model



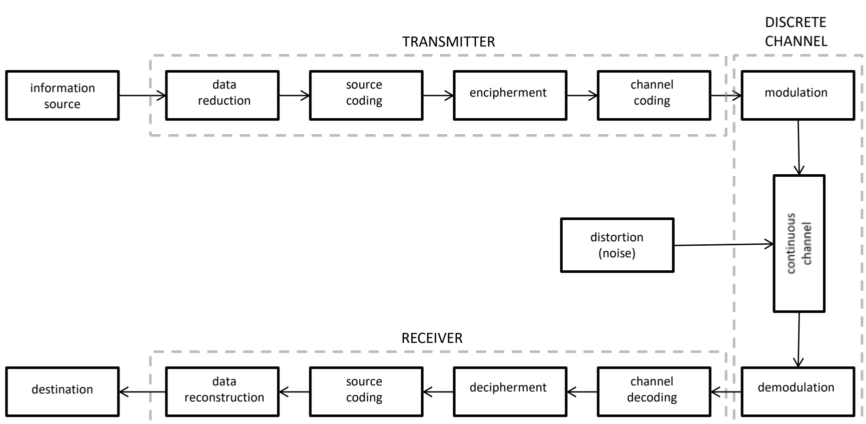
Detailed Communication Model



Detailed Communication Model



Detailed Communication Model



Error Correcting Codes

- Hamming Codes (1950)
- Linear Codes
- Low Density Parity Codes (1960)
- Convolutional Codes
- Turbo Codes (1993)

Applications of Error Correcting Codes

- Satellite communications
- Spacecraft/space probe communications: Mariner 4 (1965), Voyager (1977-), Cassini (1990s), Mars Rovers (1997-)

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- Satellite communications
- Spacecraft/space probe communications: Mariner 4 (1965), Voyager (1977-), Cassini (1990s), Mars Rovers (1997-)
- CD and DVD Players, etc.
- Internet & Web communications (Ethernet, IP, IPv6, TCP, UDP, etc)
- Data storage
- ECC Memory (DRAM)
- Etc. etc. etc.

Error Correcting Codes: Checksum

- ISBN: 0-691-12418-3
- 1*0+2*6+3*9+4*1+5*1+6*2+7*4+8*1+9*8
  = 168 mod 11 = 3
- This is a staircase checksum

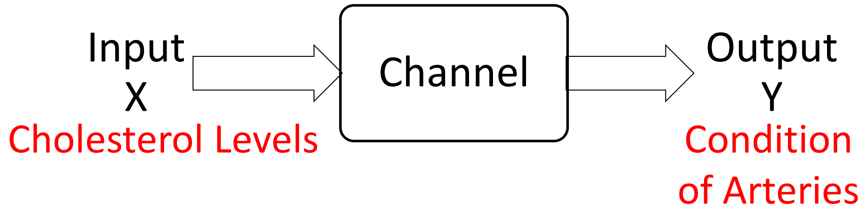
ISBN Checksum Exercise

- For the 13-digit ISBN (e.g. 978-0-252-72546-3, for *Shannon & Weaver's The Mathematical Theory of Communication*)

  Each digit, from left to right, is alternately multiplied by 1 or 3, then those products are summed modulo 10 to give a value, R ranging from 0 to 9.
  Checksum digit is 10 – R.

  Thus, 978-0-19-955137-?
  (*Floridi's Information: A Very Short Introduction*)

Information Channel is a General Model



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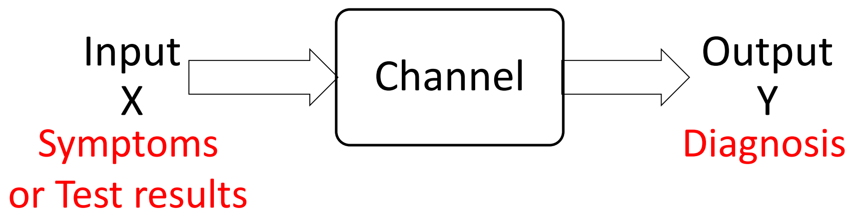

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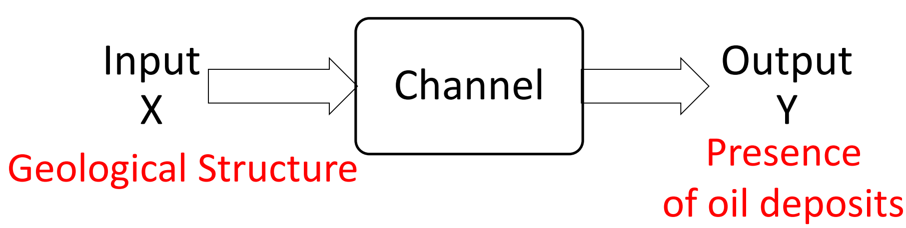

Information Channel is a General Model

MTC: Summary

- Information
- Entropy
- Source Coding Theorem
- Redundancy
- Compression
- Huffman Encoding
- Lempel-Ziv Coding

- Channel
- Conditional Entropy
- Joint Entropy
- Mutual Information
- Channel Capacity
- Shannon's Second Theorem
- Error Correction Codes

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