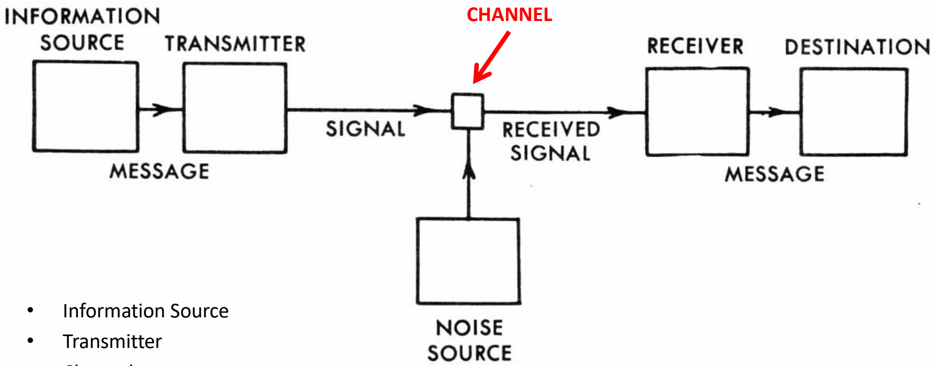
Introduction to Information Theory

Part 2

A General Communication System



- Information Source
- Transmitter
- Channel
- Receiver
- Destination

Definition of Information

- Information is quantified using probabilities.
- Given a finite set of possible messages, associate a probability with each message.
- A message with low probability represents more information than one with high probability.

Definition of Information:

$$\mathrm{I} = \log_2\left(\frac{1}{\mathrm{p}}\right) = -\log_2(\mathrm{p})$$

Where p is the probability of the message

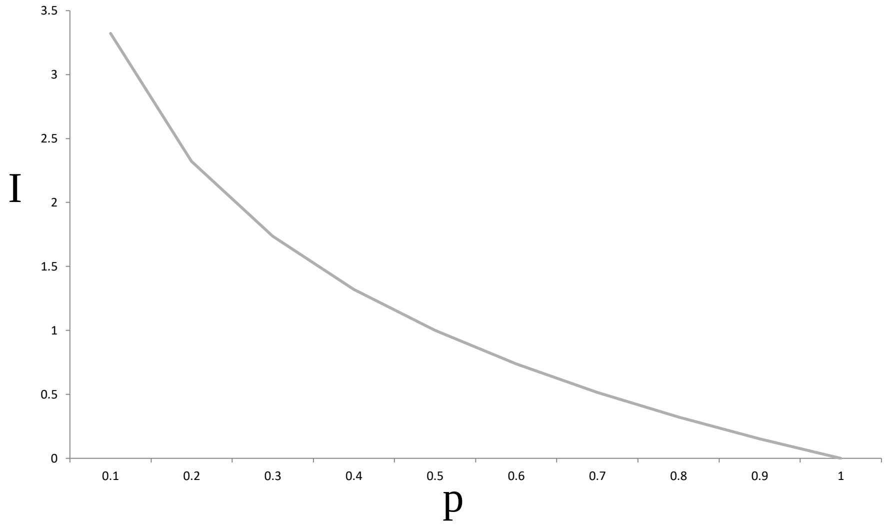
Base 2 is used for the logarithm so I is measured in **bits**

Example: Information in a coin flip

$$\mathrm{p(HEADS)} = \frac{1}{2}$$

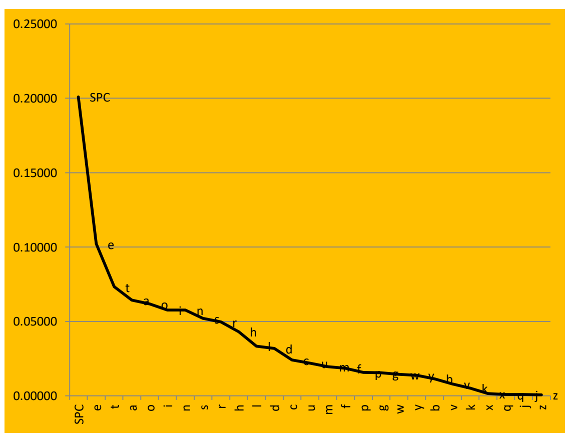
$$\mathrm{I} = -\log_2\left(\frac{1}{2}\right) = 1\ \mathrm{bit}$$

Information Content



Example: Text Analysis

a	0.06428
b	0.01147
c	0.02413
d	0.03188
e	0.10210
f	0.01842
g	0.01543
h	0.04313
i	0.05767
j	0.00082
k	0.00514
l	0.03338
m	0.01959
n	0.05761
o	0.06179
p	0.01571
q	0.00084
r	0.04973
s	0.05199
t	0.07327
u	0.02201
v	0.00800
w	0.01439
x	0.00162
y	0.01387
z	0.00077
SPC	0.20096



Example Text Analysis

Letter	Freq.	I
a	0.06428	3.95951
b	0.01147	6.44597
c	0.02413	5.37297
d	0.03188	4.97116
e	0.10210	3.29188
f	0.01842	5.76293
g	0.01543	6.01840
h	0.04313	4.53514
i	0.05767	4.11611
j	0.00082	10.24909
k	0.00514	7.60474
l	0.03338	4.90474
m	0.01959	5.67385
n	0.05761	4.11743
o	0.06179	4.01654
p	0.01571	5.99226
q	0.00084	10.21486
r	0.04973	4.32981
s	0.05199	4.26552
t	0.07327	3.77056
u	0.02201	5.50592
v	0.00800	6.96640
w	0.01439	6.11899
x	0.00162	9.26697
y	0.01387	6.17152
z	0.00077	10.34877
SPC	0.20096	2.31502

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Informattion, $I = \log(1/p)$

Some properties of I

1. $I(p) \geq 0$
   Information is non-negative.

2. $I(p_1 * p_2) = I(p_1) + I(p_1)$
   Information we get from observing two independent events occurring is the sum of two information(s).

3. $I(p)$ is monotonic and continuous in p
   Slight changes in probability incur slight changes in information.

4. $I(p = 1) = 0$
   We get zero information from an event whose probability is 1.

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Entropy

- Information (I) is associated with known events/messages that have occurred.
- Entropy is a measure of information we expect to receive in the future.
- It is the average information w.r.to all possible outcomes

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Entropy

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- Entropy is a measure of information we expect to receive in the future.
- It is the average information w.r.to all possible outcomes

$$p_1 I_1 + p_2 I_2 + p_3 I_3 + \cdots$$

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Definition of Entropy

- Information (I) is associated with known events/messages
- Entropy (H) is the average information w.r.to all possible outcomes

$$H(X) = \sum_x p_x \log \frac{1}{p_x}$$

- H is also measured in bits

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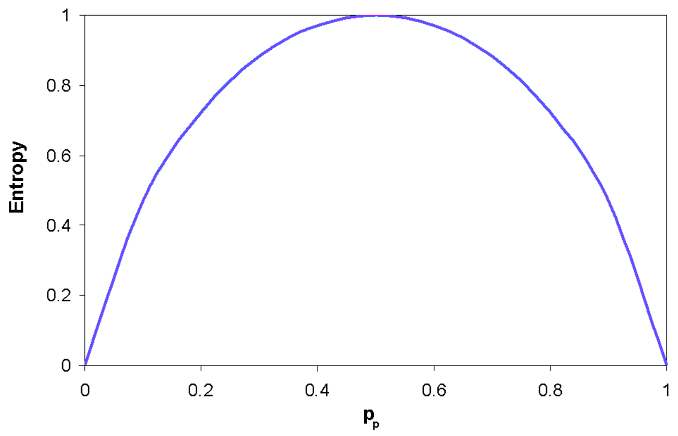
Entropy (2 outcomes: $p, 1 - p$)

$$H(p) = p \log\left(\frac{1}{p}\right) + (1 - p)\log(\frac{1}{1 - p})$$



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Examples: Weather

- 2 event source: (Sunny, Cloudy)

$$p_{sunny} = \frac{7}{8}$$

$$p_{cloudy} = \frac{1}{8}$$

- 3 – event source (Sunny, Cloudy, Precip)

  Bryn Mawr: (207, 119, 39)

Let us compute this.

Example Text Analysis

Letter	Freq.	I, h(pi)
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d	0.03188	4.97116
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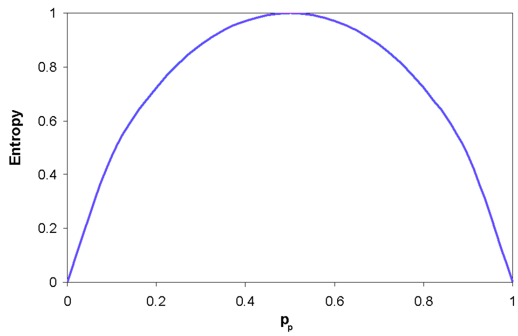
$$H(X) = \sum_{x} p_x \log \frac{1}{p_x} = 4.047$$

Entropy: Properties



$H(X) \geq 0$

$H(X_n) \leq \log_2(n)$
Entropy is maximized if p is uniform.

$H(S, T) = H(S) + H(T)$
Additive Property

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Entropy of S^n

- S is a source with k independent events and H(S) = e
  e.g. S = [H, T]
  H, H, T, H, T, H, …
  H(S) = 1

- $S^2$is a source consisting of two observations of events from S
  e.g. S = [H, T]
  TH, TT, HH, HH, TT, HT, …

  then, $H(S^2) = 2\,H(S)$

- In general,

$$H(S^n) = n\,H(S)$$

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Entropy of things…

- Entropy of English text is approx 1.5 bits/letter
- Entropy of the human genome <= 2 bits
- Entropy of a black hole is ¼ of the area of the outer event horizon.
- Value of information in economics is defined in terms of entropy. E.g. Scarcity

$$V(X) = \sum_{i=1}^{n} p_i(-\log_b(p_i))$$

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bit versus bit **- Two meanings**

- bit as measure of information/entropy
- bit as a binary digit

e.g.	01001101	is six bits long
weather	01001101	8 days sunny/cloudy(0/1)
		information is less than 8 bits

Information represented as decimal digits
log(10) = 3.32, thus the string 32767 has 6 * 3.32 = 19.92 bits of information

26 letter-alphabet has average information, log(26) = 4.7 bits

- Bits needed to store n symbols matches entropy in bits only when all symbols are equally likely and are mutually independent.

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So, what is Entropy good for???

- Provides the foundation for techniques for
  - Compression
  - Searching in data
  - Encryption
  - Correcting communication errors
  - Extracting information from data
  - Economic value of information
  - Biological information
  - Quantum information
  - Etc.

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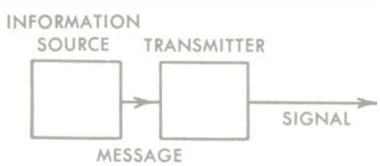

Coding

- An information source, S has m events
- Thus, m symbols are to be transmitted: $s_1, s_2, s_3, \dots, s_m$

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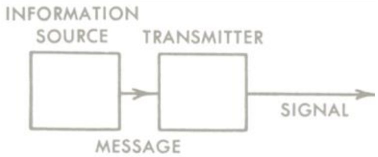

Coding

- An information source, S has m events
- Thus, m symbols are to be transmitted: $s_1, s_2, s_3, \dots, s_m$

- A **code** is an assignment of **codewords** to source symbols
- Codewords are made up of characters from a **code alphabet**

e.g. $S = \{SUNNY, PRECIP, RAINY\}$

code alphabet = {0, 1}

Code:

SUNNY	→	0
PRECIP	→	01
RAINY	→	010

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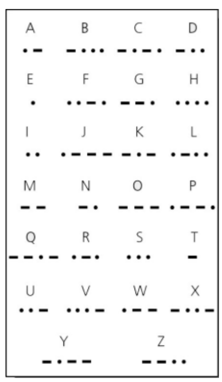
Coding: Basics

- Events of an information source: $s_1, s_2, \dots, s_n$

- A **code** is made up of **codewords** from a **code alphabet** (e.g. {0, 1}, {., -}, etc.)



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Coding: Basics

- **Block code**: When all codes have the same length. For example, ASCII, Unicode, etc.
- **Average Word Length**: $L = \sum_{i=1}^{m} p_i l_i$
- **Singular** (not unique) codes
- **Nonsingular** (unique) codes
- **instantaneous** codes

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Coding: Basics

- **Block code**: When all codes have the same length. For example, ASCII ($l = 8$).
- **Average Word Length**: $L = \sum_{i=1}^{m} p_i l_i$

Code length is important!

Short codewords preferred to long ones.

Useless code!

- **Singular** (not unique) codes
- **Nonsingular** (unique) codes
- **instantaneous** codes

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Example Code

Source Symbol	Singular Code	Nonsingular Code
A	00	0
B	10	10
C	01	00
D	10	01

Example Code

Source Symbol	Singular Code	Nonsingular Code
A	00	0
B	10	10
C	01	00
D	10	01

In practice, nonsingularity is not sufficient.

e.g. receiver gets: 0010

ADA?
CD?
AAB?

Nonsingular, Instantaneous, Block Code

Source Symbol	Nonsingular Code
A	00
B	01
C	10
D	10

e.g. receiver gets: 01101100

Comma Codes & Capital Codes

Source Symbol	Comma Code	Capital Code
A	0	0
B	10	01
C	110	011
D	1110	0111

One of these is instantaneous.

e.g. receiver gets: 01011100

receiver gets: 00101110

Example

Symbol	p	Codeword
A	0.3	00
B	0.2	10
C	0.2	11
D	0.2	010
E	0.1	011

$$\mathrm{L} = (0.3 * 2) + (0.2 * 2) + (0.2 * 2) + (0.2 * 3) + (0.1 * 3) = 2.3$$

Example

Symbol	p	Codeword
A	0.3	00
B	0.2	10
C	0.2	11
D	0.2	010
E	0.1	011

$$\mathrm{L} = (0.3 * 2) + (0.2 * 2) + (0.2 * 2) + (0.2 * 3) + (0.1 * 3) = 2.3$$

$$\mathrm{H} = 0.3 \log\left(\frac{1}{0.3}\right) + 0.2 \log\left(\frac{1}{0.2}\right) * 3 + 0.1 \log\left(\frac{1}{0.1}\right) = 2.246$$

Example

Symbol	p	Codeword
A	0.3	00
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Is there a relationship between L and H?

Average Code Length & Entropy

- Average length bounds: $\mathrm{H} \leq \mathrm{L} < \mathrm{H} + 1$
- Grouping n symbols together:

$$H(S^n) \leq L \leq H(S^n) + 1$$

Average Code Length & Entropy

- Average length bounds: $\mathrm{H} \leq \mathrm{L} < \mathrm{H} + 1$
- Grouping n symbols together:

$$\mathrm{H}(\mathrm{S}^{\mathrm{n}}) \leq \mathrm{L} < \mathrm{H}(\mathrm{S}^{\mathrm{n}}) + 1$$

$$\mathrm{nH}(\mathrm{S}) \leq \mathrm{L} < \mathrm{nH}(\mathrm{S}) + 1$$

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Average Code Length & Entropy

- Average length bounds: $\mathrm{H} \leq \mathrm{L} < \mathrm{H} + 1$
- Grouping n symbols together:

$$H(S^n) \leq L \leq H(S^n) + 1$$

$$nH(S) \leq L \leq nH(S) + 1$$

$$H(S) \leq \frac{L}{n} \leq H(S) + \frac{1}{n}$$

This is for instantaneous binary codes.

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Average Code Length & Entropy

- Average length bounds: $\mathrm{H} \leq \mathrm{L} < \mathrm{H} + 1$
- Grouping n symbols together:

$$H(S^n) \leq L \leq H(S^n) + 1$$

$$nH(S) \leq L \leq nH(S) + 1$$

$$H(S) \leq \frac{L}{n} \leq H(S) + \frac{1}{n}$$

$$\lim_{\mathrm{n}\to\infty} \frac{\mathrm{L_n}}{\mathrm{n}} = \mathrm{H}$$

H is the entropy of source S
n is the length of symbol sequences
$\mathrm{L_n}$ is the avg. length of codewords

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Shannon's First Theorem

- By coding sequences of independent symbols (in S^n), it is possible to construct codes such that

$$\lim_{\mathrm{n}\to\infty} \frac{\mathrm{L_n}}{\mathrm{n}} = \mathrm{H}$$

The price paid for such improvement is increased coding complexity (due to increased n) and increased delay in coding.

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Question

- Is there a **coding algorithm** that produces codes such that it achieves Shannon limit?

  $L = H$?

Yes!

Huffman's algorithm (**Huffman Coding**) produces a code with average length L as close as possible to source code entropy, H.

Data Compression: Huffman Coding

A	0.3
B	0.2
C	0.2
D	0.2
E	0.1

Huffman Coding: Reduction Phase

A	0.3	0.3	0.4	0.6
B	0.2	0.3	0.3	0.4
C	0.2	0.2	0.3	
D	0.2	0.2		
E	0.1			

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Huffman Coding: SplittingPhase

A	0.3 00	0.3 00	0.4 1	0.6 0
B	0.2 10	0.3 01	0.3 00	0.4 1
C	0.2 11	0.2 10	0.3 01	
D	0.2 010	0.2 11		
E	0.1 011			

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Huffman Coding: SplittingPhase

A	0.3 00	0.3 00	0.4 1	0.6 0
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D	0.2 010	0.2 11		
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$$H = 2.246$$

$$L = (0.3 * 2) + (0.2 * 2) + (0.2 * 2) + (0.2 * 3) + (0.1 * 3) = 2.3$$

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Huffman Coding: Text Compression

text $\rightarrow$ Compress $\rightarrow$ compressed text

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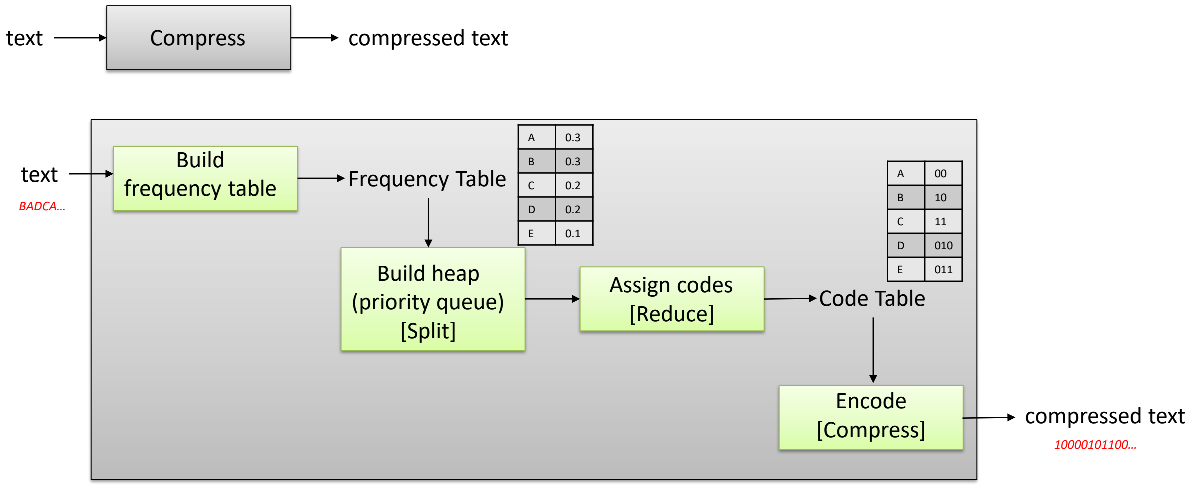
Huffman Coding: Text Compression



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Huffman Coding: Text Compression

For English text with 27 characters (A, .., Z, SPC)

$$H(T) = log_2(27) = 4.755$$

Instead of using 8-bit ASCII, we can encode using Huffman codes, with L <= 4.7 and get 50% compression.

In fact, Entropy of English texts is much less than 4, since all characters are not uniformly distributed.

In practice, compression rates of 60% are typical.

Other Coding Schemes

- Huffman Coding
- Lempel-Ziv (LZ77)
  ZIP, PKSip, PNG, gzip, …
- Lempel-Ziv (LZ78)
- Lempel-Ziv-Welch (LZW, 1984)
  compress, GIF, PDF, etc.
- Prediction Methods
  JPEG (lossless & lossy)
- Perceptual Coding
  MPEG, MPEG1, MP3, etc.

Entropy & Coding

- Use short codes for highly likely events. This shortens the average length of coded messages.
- Code several events at a time. Provides greater flexibility in code design.

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References

- Eugene Chiu, Jocelyn Lin, Brok Mcferron, Noshirwan Petigara, Satwiksai Seshasai: *Mathematical Theory of Claude Shannon: A study of the style and context of his work up to the genesis of information theory. MIT* 6.933J / STS.420J The Structure of Engineering Revolutions
- Luciano Floridi, 2010: *Information: A Very Short Introduction*, Oxford University Press, 2011.
- Luciano Floridi, 2011: *The Philosophy of Information*, Oxford University Press, 2011.
- James Gleick, 2011: *The Information: A History, A Theory, A Flood*, Pantheon Books, 2011.
- Zhandong Liu , Santosh S Venkatesh and Carlo C Maley, 2008: *Sequence space coverage, entropy of genomes and the potential to detect non-human DNA in human samples*, *BMC Genomics* 2008, **9:**509
- David Luenberger, 2006: *Information Science*, Princeton University Press, 2006.
- David J.C. MacKay, 2003: *Information Theory, Inference, and Learning Algorithms*, Cambridge University Press, 2003.
- Claude Shannon & Warren Weaver, 1949: *The Mathematical Theory of Communication*, University of Illinois Press, 1949.
- W. N. Francis and H. Kucera: *Brown University Standard Corpus of Present-Day American English*, Brown University, 1967.

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