Introduction to Information Theory Part 1

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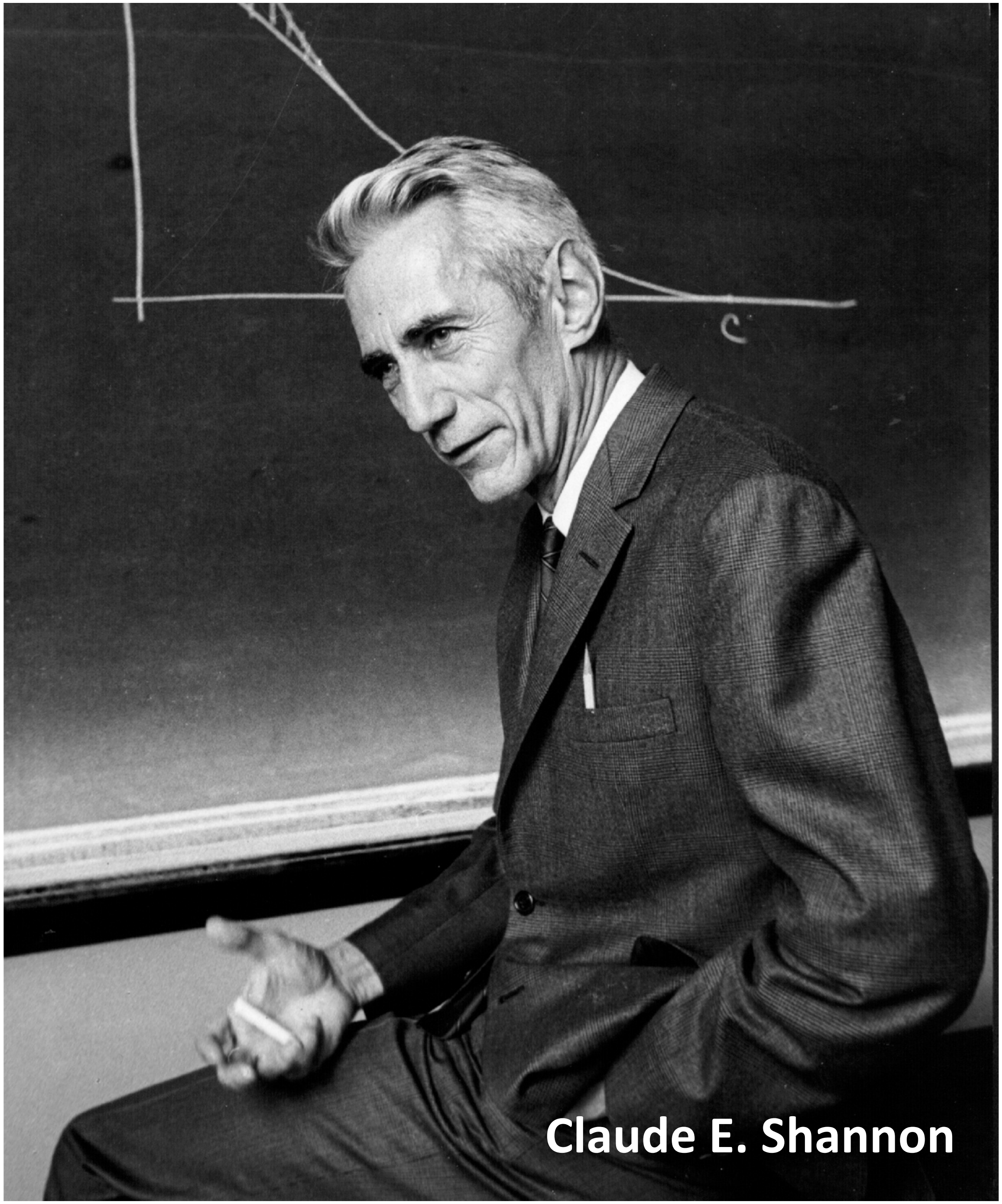
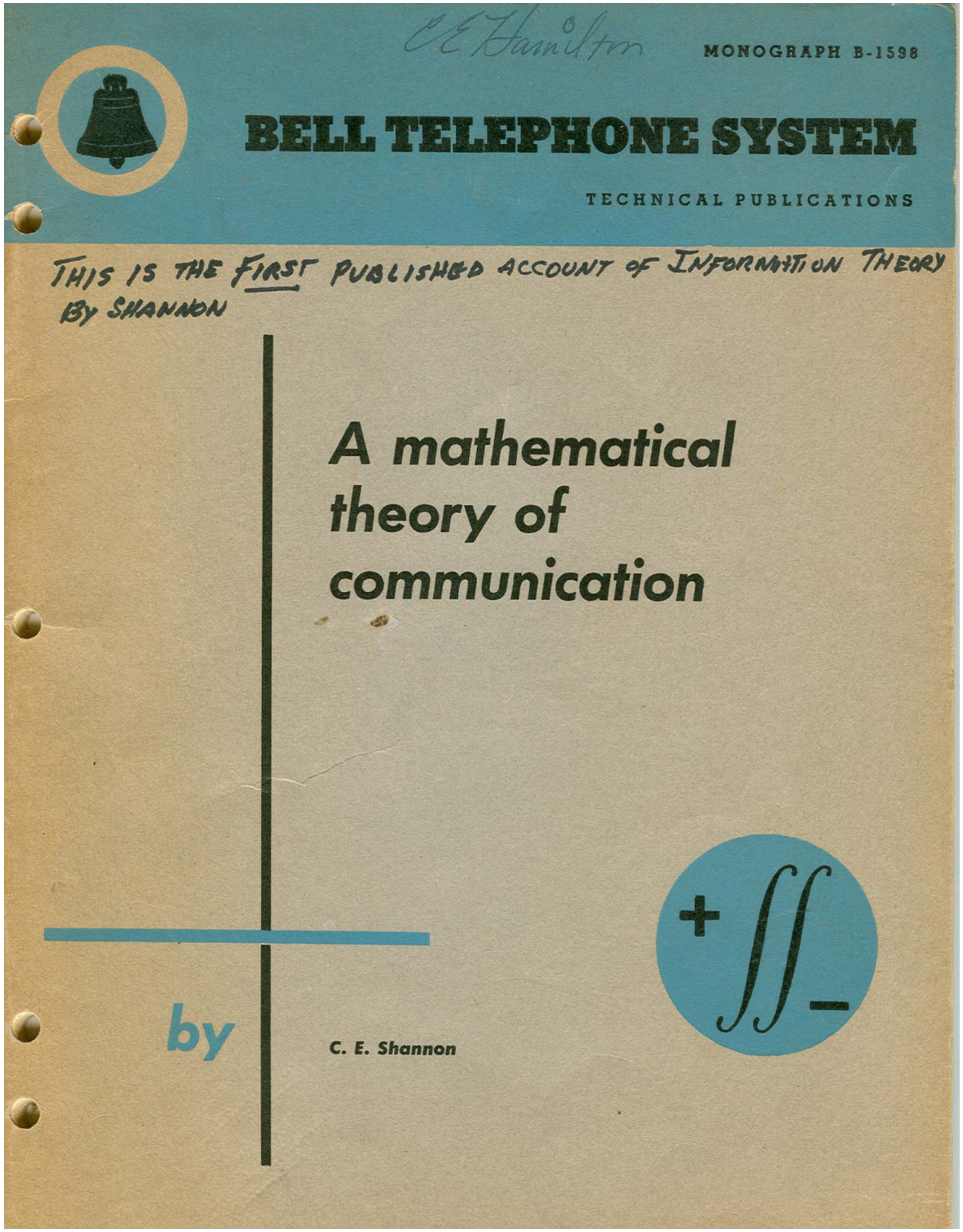


Claude E. Shannon



C E Hamilton

MONOGRAPH B-1598

BELL TELEPHONE SYSTEM

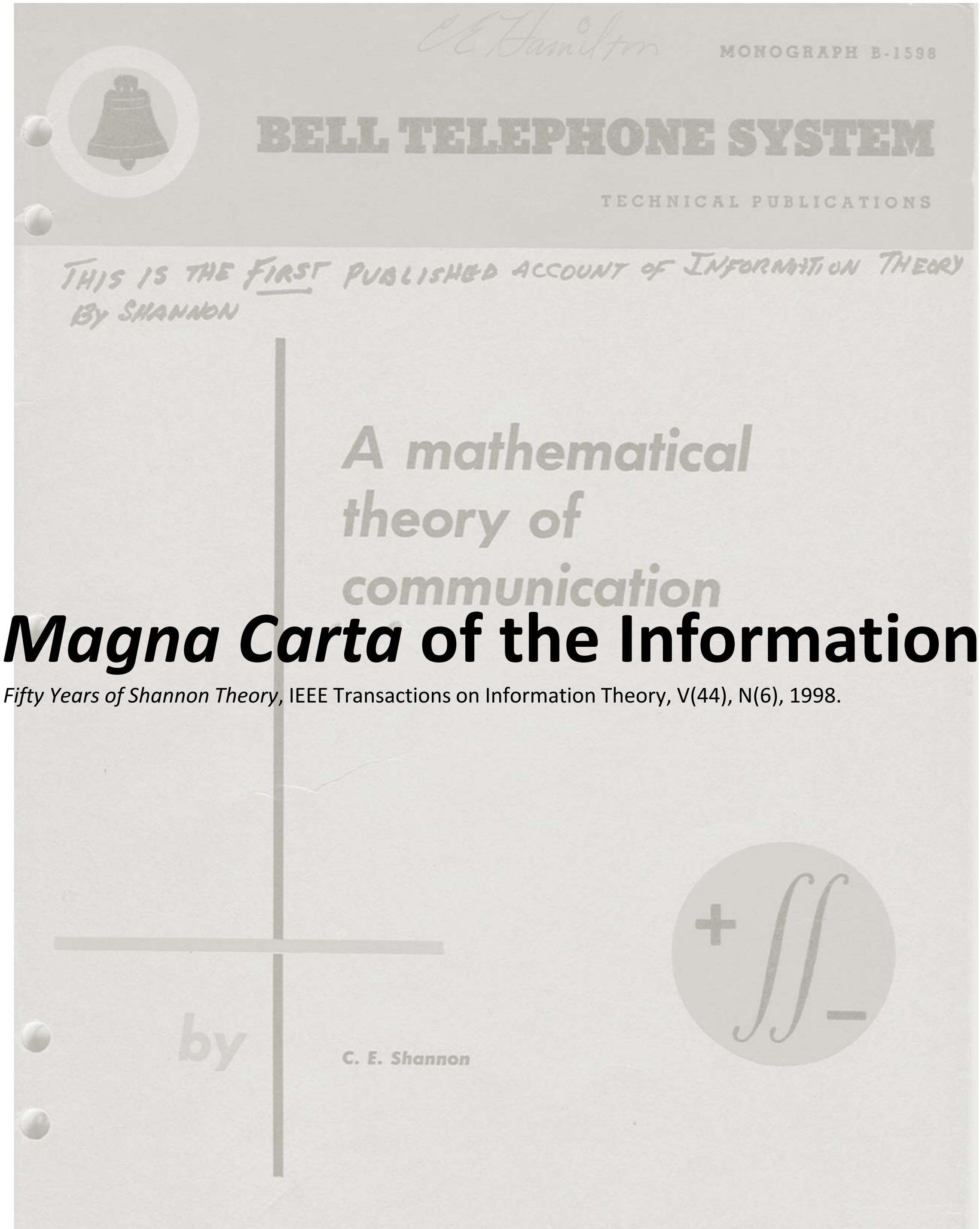
TECHNICAL PUBLICATIONS

THIS IS THE FIRST PUBLISHED ACCOUNT OF INFORMATION THEORY BY SHANNON

A mathematical theory of communication

by

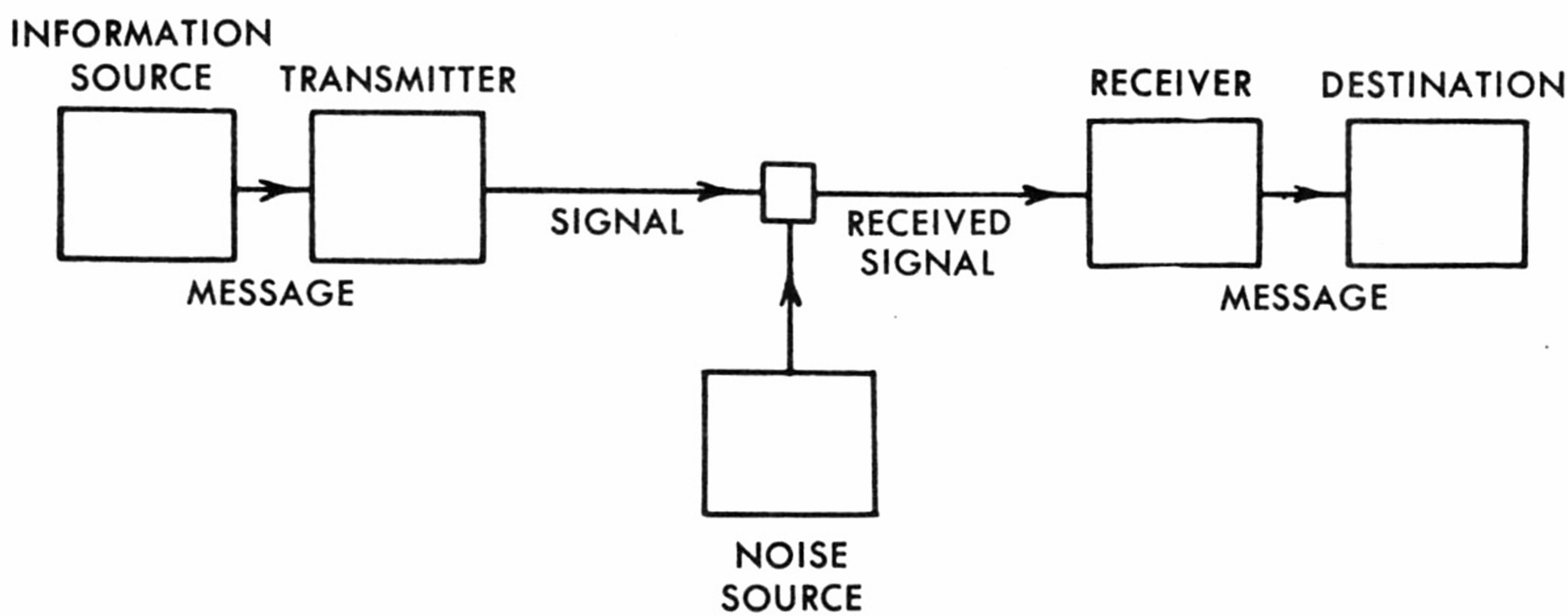
C. E. Shannon



The *Magna Carta* of the Information Age

Sergio Verdu, *Fifty Years of Shannon Theory*, IEEE Transactions on Information Theory, V(44), N(6), 1998.

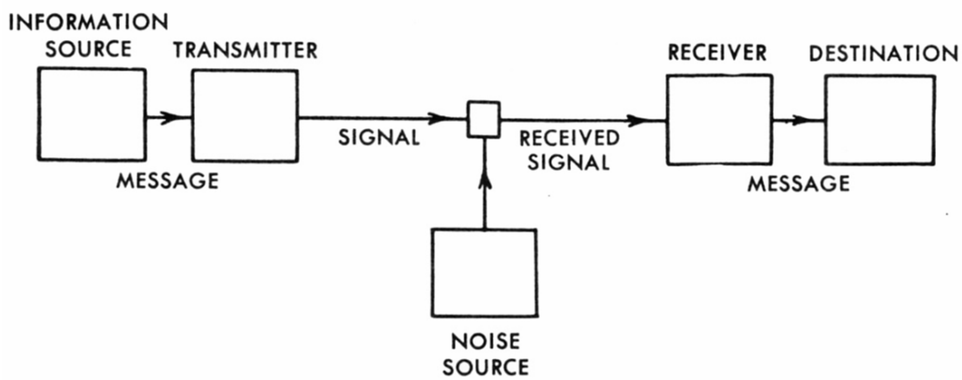
A General Communication System



<u>Figure 1.</u>

A Mathematical Theory of Communication, by Claude E. Shannon, *The Bell System Technical Journal,* Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Shannon's Information Theory



- Conceptualization of information & modeling of information sources
- Sending of information across the channel:
  - ➢ What are the limits on the amount of information that can be sent?
  - ➢ What is the effect of noise on this communication.

A General Communication System

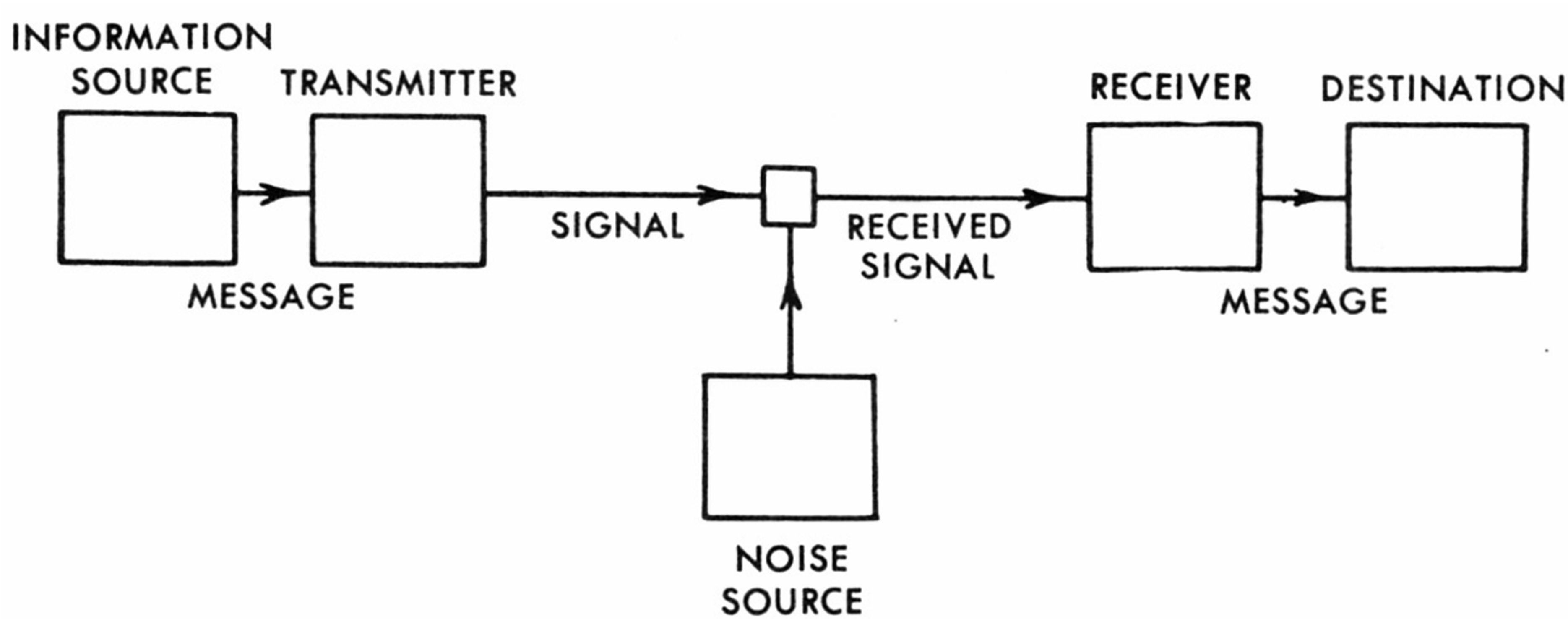
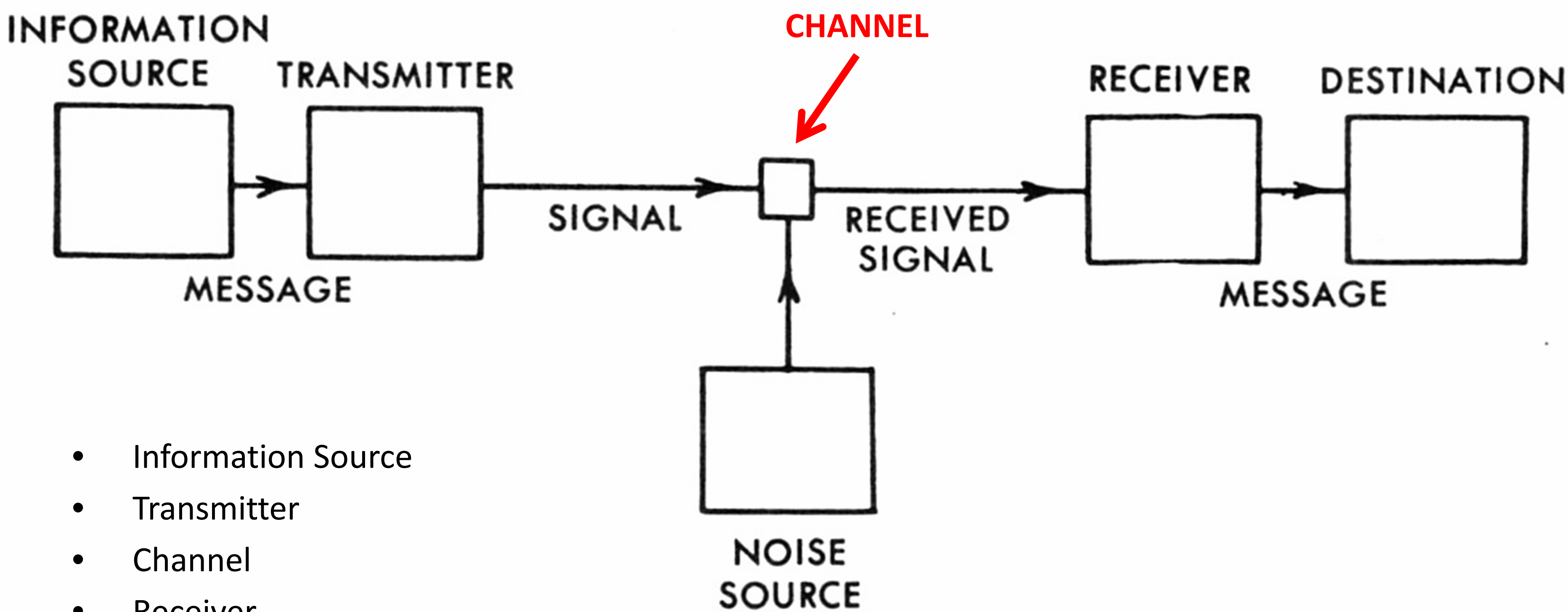


Figure 1.

A Mathematical Theory of Communication, by
Claude E. Shannon, *The Bell System Technical Journal,*
Vol. 27, pp. 379–423, 623–656, July, October, 1948.

A General Communication System



- Information Source
- Transmitter
- Channel
- Receiver
- Destination

A Mathematical Theory of Communication, by
Claude E. Shannon, *The Bell System Technical Journal,*
Vol. 27, pp. 379–423, 623–656, July, October, 1948.

How much information?

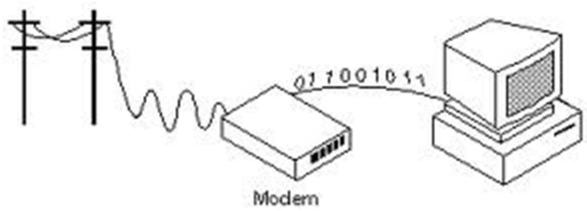
ᛊᚲᛁᛖᚾᚲᛖ ᛟᚠ ᛁᚾᚠᛟᚱᛗᚨᛏᛁᛟᚾ

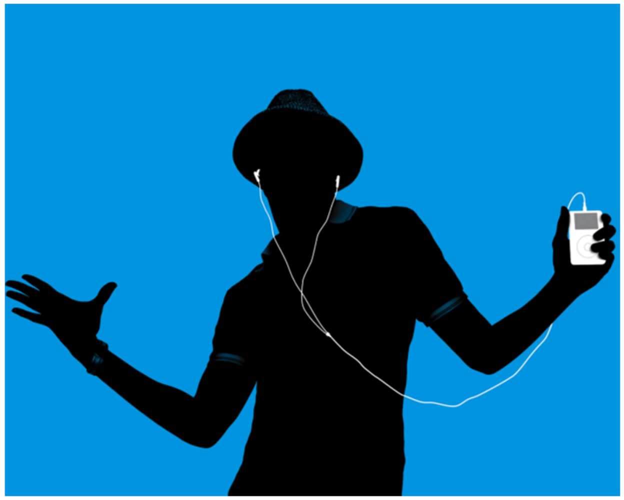
ᚠ	ᚢ	ᚦ	ᚨ	ᚱ	ᚲ	ᚷ	ᚹ
f	u	þ	a	r	k	g	w
fehu	ūruz	þurisaz	ansuz	raiþō	kaunaz	gebō	wunjō
wealth	aurochs	giant	god	riding	ulcer	gift	joy
ᚺ	ᚾ	ᛁ	ᛃ	ᛇ	ᛈ	ᛉ	ᛊ
h	n	i	j	ï	p	z	s
hagalaz	nauþiz	isa	jera	eihwaz	perþ	algiz	sōwulō
hail	need/hardship	ice	year/harvest	yew tree	luck	sedge (?)	sun
ᛏ	ᛒ	ᛖ	ᛗ	ᛚ	ᛜ	ᛞ	ᛟ
t	b	e	m	l	ng	d	o
teiwaz	berkana	ehwaz	mannaz	laguz	inguz	đagaz	ōþila
the god Tyr	birch twig	horse	man	water	the god Ing	day	inherited land

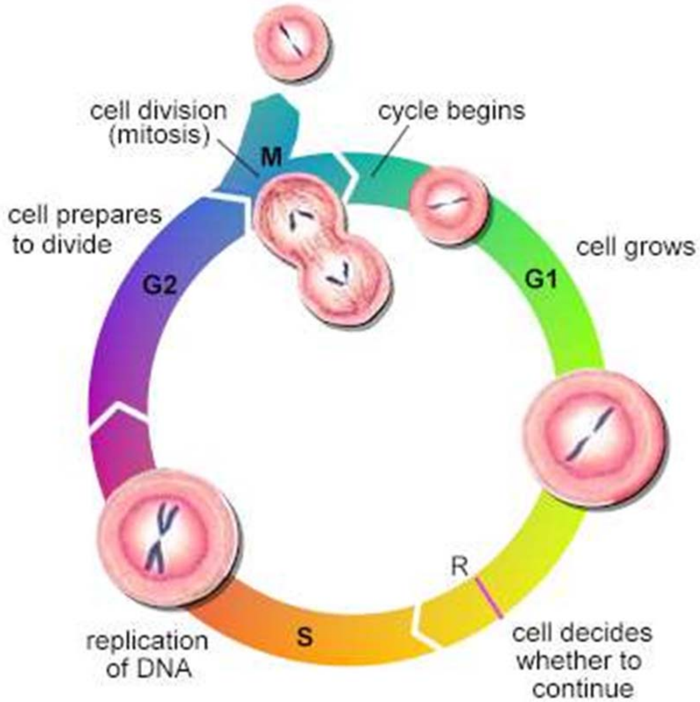
“The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.”

A Mathematical Theory of Communication, by
Claude E. Shannon, *The Bell System Technical Journal,*
Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Examples of Communication Systems

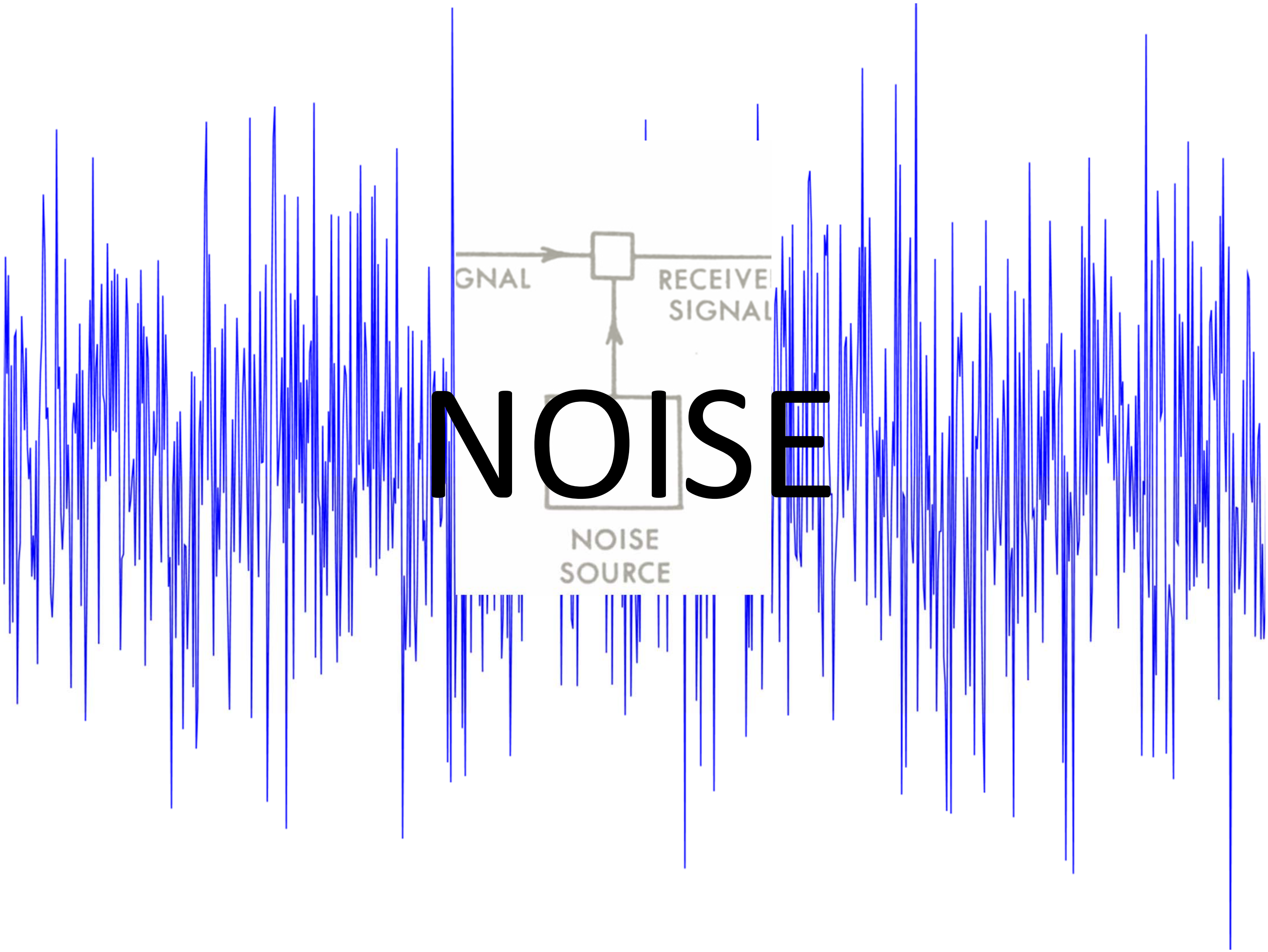




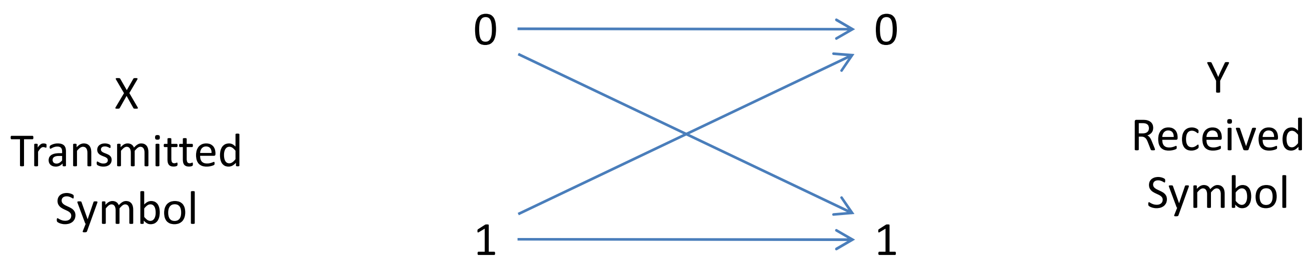


Perfect Communication (*Noiseless Channel*)

NOISE

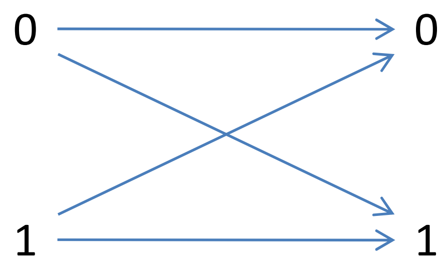


Motivating Noise…



Motivating Noise…

X
Transmitted
Symbol



Y
Received
Symbol

P(Y=1|X=0) = f

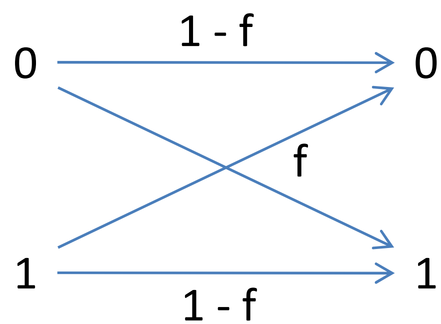
P(Y=0|X=1) = f

P(Y=0|X=0) = 1 - f

P(Y=1|X=1) = 1 - f

Motivating Noise…

f = 0.1, n = ~10,000



Question

How can we achieve perfect communication over an imperfect, noisy communication channel?

- Use more reliable components;
- Stabilize the environment;
- Use larger areas;
- Use power/cooling to reduce thermal noise.

These are all costly solutions.

Alternately…

How can we achieve perfect communication over an imperfect, noisy communication channel?

- Accept that there will be noise
- Add error detection and correction
- Introduce the concepts of ENCODER/DECODER

Information Theory
Theoretical limitations of such systems

Coding Theory
Creation of practical encoding/decoding systems

Alternately…

How can we achieve perfect communication over an imperfect, noisy communication channel?

- Accept that there will be noise
- Add error detection and correction
- Introduce the concepts of ENCODER/DECODER

REDUNDANCY IS KEY!

Information Theory
Theoretical limitations of such systems

Coding Theory
Creation of practical encoding/decoding systems

Shannon's Insight

High Reliability → Low Transmission Rate

I.e. Perfect reliability → Zero Transmission Rate

For a given level of noise there is an associated rate of transmission that can be achieved with arbitrarily good reliability.

e.g. Sending a lone T versus THIS…

Meaning? What meaning?

“Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities.

These semantic aspects of communication are irrelevant to the engineering problem.”

A Mathematical Theory of Communication, by
Claude E. Shannon, *The Bell System Technical Journal,*
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What is information?

- Just the physical aspects…Shannon, 1948
- The General Definition of Information…Floridi, 2010.

GDI) σ is an instance of information, understood as semantic content, if and only if:

GDI.1) σ consists of *n data,* for $n \geq 1$;

GDI.2) the data are *well formed;*

GDI.3) the well-formed data are *meaningful.*

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Shannon Information is…

- Uncertainty

  Can be measured by counting the number of possible messages.

- Surprise

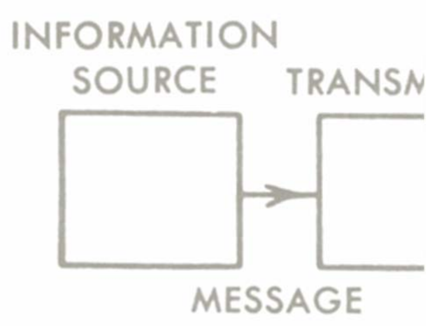
  Some messages are more likely than others.

- Difficult

  What is significant is the difficulty of transmitting the message from one point to the other.

- Entropy

  A fundamental measure of information.



Information Source

- An information source generates a finite number of messages (or symbols).
- Information is quantified using probabilities.
- Given a finite set of possible messages, associate a probability with each message.
- A message with low probability represents more information than one with high probability.

Understanding Information

It is sunny in California today!

Information: Definition

- Information is quantified using probabilities.
- Given a finite set of possible messages, associate a probability with each message.
- A message with low probability represents more information than one with high probability.

Definition of Information:

$$\boldsymbol{I(p) = log\left(\frac{1}{p}\right) = -log(p)}$$

Where $\boldsymbol{p}$ is the probability of the message
Base 2 is used for the logarithm so $\boldsymbol{I}$ is measured in **bits**
Trits for base 3, **nats** for base e, **Hartleys** for base 10…

$$\boldsymbol{I(p) = \log\left(1/p\right) = -\log(p)}$$

Some properties of I

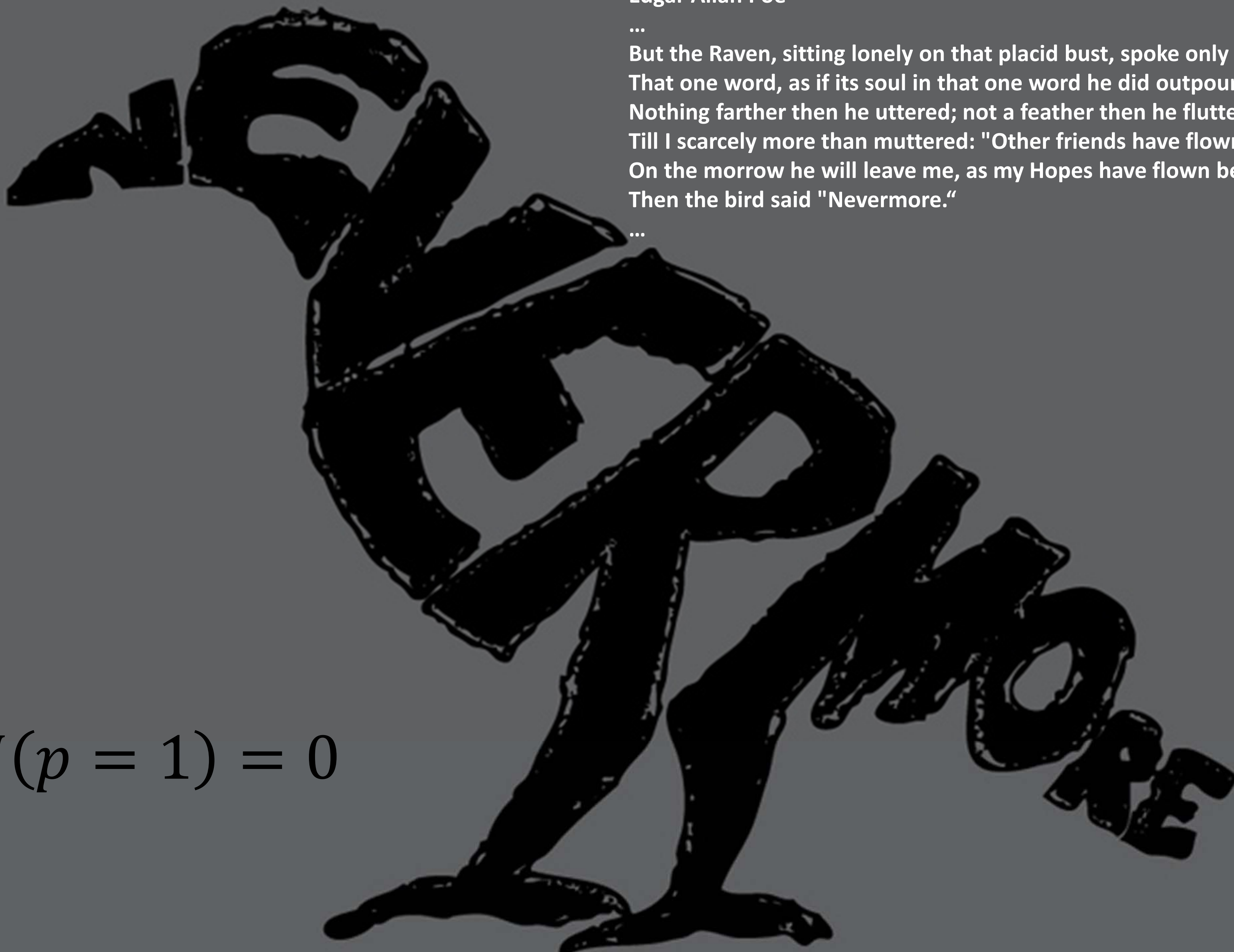
1. $I(p) \geq 0$
   Information is non-negative.

2. $I(p_1 * p_2) = I(p_1) + I(p_1)$
   Information we get from observing two independent events occurring is the sum of two information(s).

3. $I(p)$ is monotonic and continuous in p
   Slight changes in probability incur slight changes in information.

4. $I(p = 1) = 0$
   We get zero information from an event whose probability is 1.

The Raven
Edgar Allan Poe
…
But the Raven, sitting lonely on that placid bust, spoke only
That one word, as if its soul in that one word he did outpour
Nothing farther then he uttered; not a feather then he fluttered--
Till I scarcely more than muttered: "Other friends have flown before--
On the morrow he will leave me, as my Hopes have flown before."
Then the bird said "Nevermore.“
…



$$I(p = 1) = 0$$

Picture From: http://adrianamelnic.ro/2012/06/theraven/

Information in a coin flip

$$P(HEADS) = ½$$

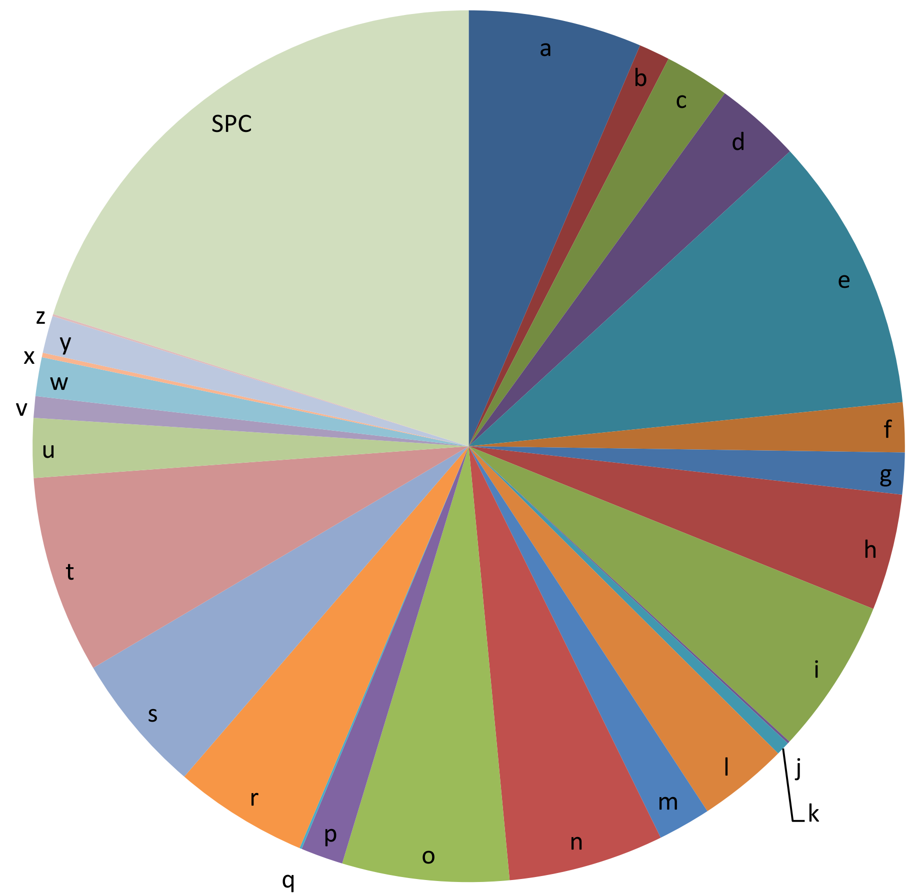
$$I = -\log(½) = 1 \text{ bit of information}$$

Given a sequence of 14 coin flips: hthhtththhttht

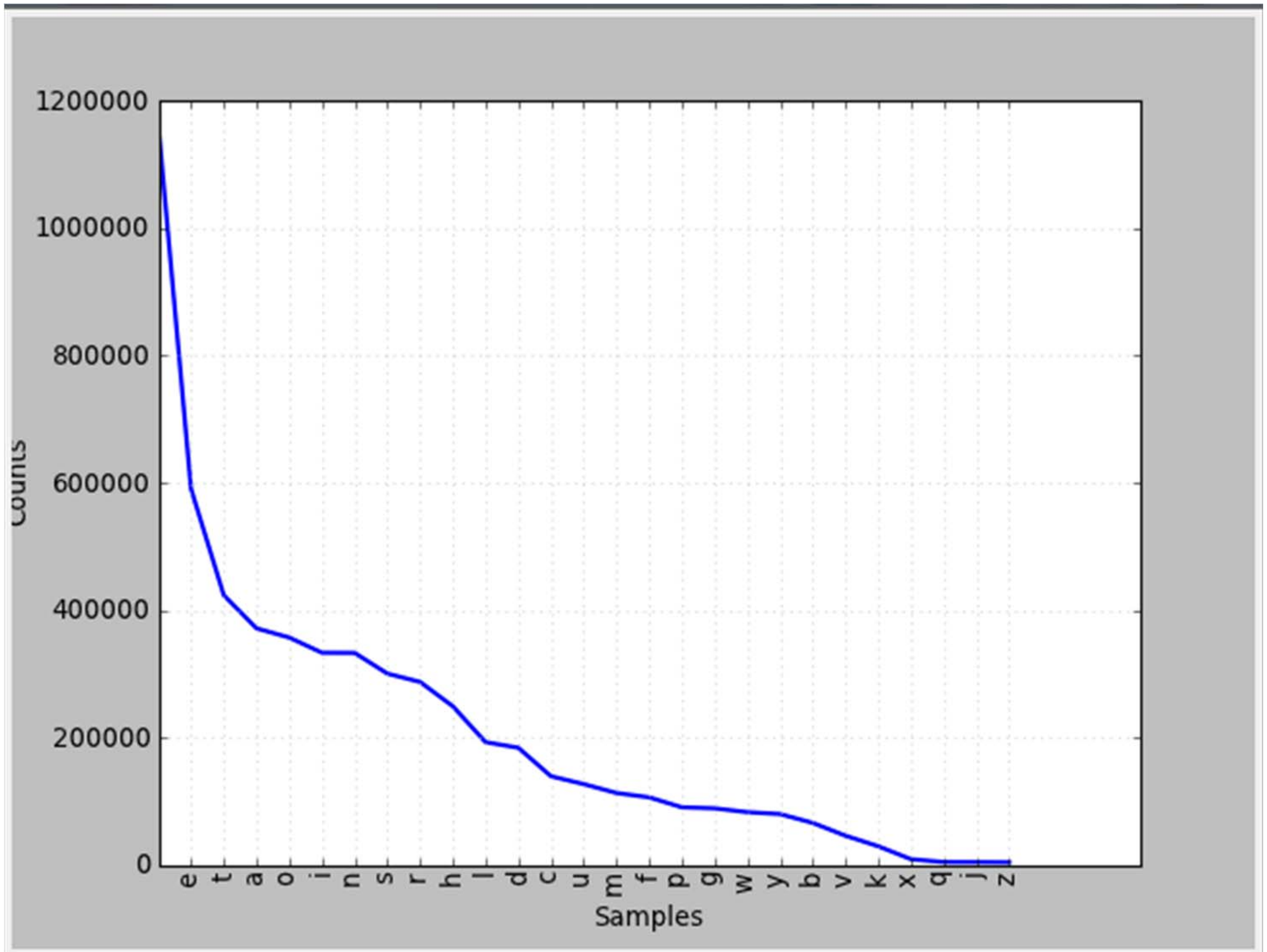
We will need 14 bits: 10110010110010

Example: Text Analysis

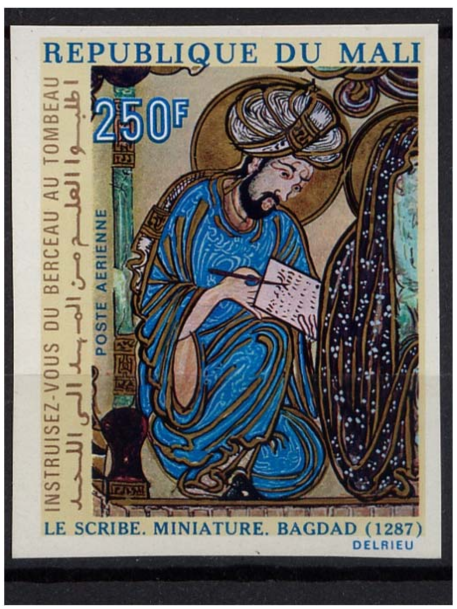
a	0.06428
b	0.01147
c	0.02413
d	0.03188
e	0.10210
f	0.01842
g	0.01543
h	0.04313
i	0.05767
j	0.00082
k	0.00514
l	0.03338
m	0.01959
n	0.05761
o	0.06179
p	0.01571
q	0.00084
r	0.04973
s	0.05199
t	0.07327
u	0.02201
v	0.00800
w	0.01439
x	0.00162
y	0.01387
z	0.00077
SPC	0.20096



Example: Text Analysis



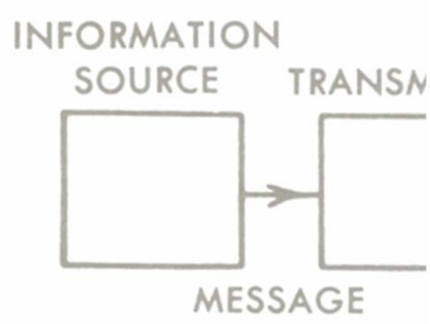
al Kindi
(9th Centry C.E.)



From: Mathematicians on Stamps: http://jeff560.tripod.com/stamps.html

Example: Text Analysis

Letter	Freq.	I
a	0.06428	3.95951
b	0.01147	6.44597
c	0.02413	5.37297
d	0.03188	4.97116
e	0.10210	3.29188
f	0.01842	5.76293
g	0.01543	6.01840
h	0.04313	4.53514
i	0.05767	4.11611
j	0.00082	10.24909
k	0.00514	7.60474
l	0.03338	4.90474
m	0.01959	5.67385
n	0.05761	4.11743
o	0.06179	4.01654
p	0.01571	5.99226
q	0.00084	10.21486
r	0.04973	4.32981
s	0.05199	4.26552
t	0.07327	3.77056
u	0.02201	5.50592
v	0.00800	6.96640
w	0.01439	6.11899
x	0.00162	9.26697
y	0.01387	6.17152
z	0.00077	10.34877
SPC	0.20096	2.31502



Information Source

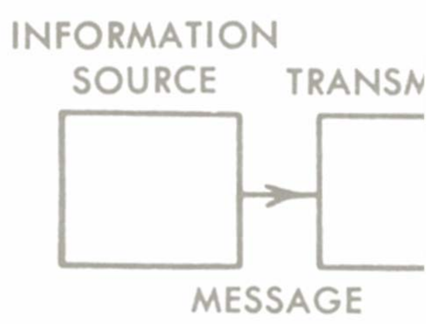
- An information source generates a finite number of messages (or symbols)

$$\{a_1, a_2, \ldots, a_n\}$$

- Source emits the symbols with probabilities

$$\mathrm{P} = \{p_1, p_2, \ldots, p_n\}$$

- Assume independence: successive symbols do not depend on past symbols.
- What is the average amount of information?



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- What is the average amount of information?

ANSWER: Entropy!

Definition of Entropy

- Information (I) is associated with known events/messages
- Entropy (H) is the average information w.r.to all possible outcomes

$$H(P) = \sum_i p_i \log \frac{1}{p_i}$$

Example: Text Analysis

Letter	Freq.	I
a	0.06428	3.95951
b	0.01147	6.44597
c	0.02413	5.37297
d	0.03188	4.97116
e	0.10210	3.29188
f	0.01842	5.76293
g	0.01543	6.01840
h	0.04313	4.53514
i	0.05767	4.11611
j	0.00082	10.24909
k	0.00514	7.60474
l	0.03338	4.90474
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x	0.00162	9.26697
y	0.01387	6.17152
z	0.00077	10.34877
SPC	0.20096	2.31502

$$H(P) = \sum_i p_i \log \frac{1}{p_i} = 4.047$$

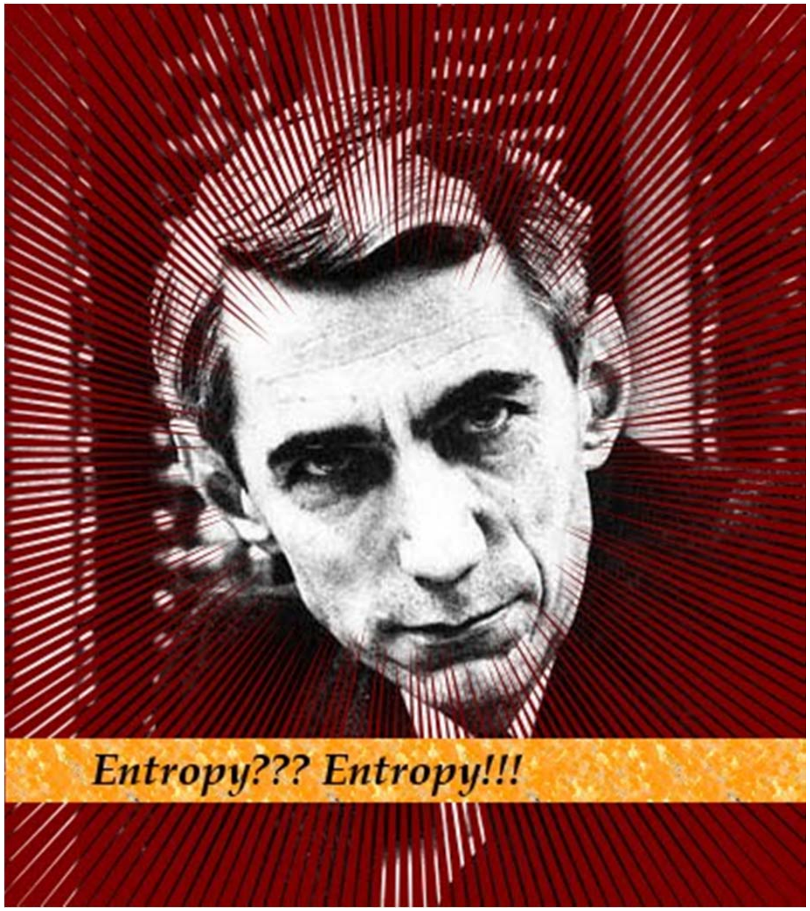
Entropy: What about it?

- Does $H(P)$ have a maximum? Where?

- Is entropy a good name for this stuff? How is it related to entropy in thermodynamics?

- How does entropy help in communication? What else can we do with it?

- Why use the letter H? ☺


Entropy??? Entropy!!!

H for Entropy?

“The enthalpy is [often] written U. V is the volume, and Z is the partition function. P and Q are the position and momentum of a particle. R is the gas constant, and of course T is temperature. W is the number of ways of configuring our system (the number of states), and we have to keep X and Y in case we need more variables. Going back to the first half of the alphabet, A, F, and G are all different kinds of free energies (the last named for Gibbs). B is a virial coefficient or a magnetic field. I will be used as a symbol for information; J and L are angular momenta. K is Kelvin, which is the proper unit of T. M is magnetization, and N is a number, possibly Avogadro's, and O is too easily confused with 0. This leaves S . . ."

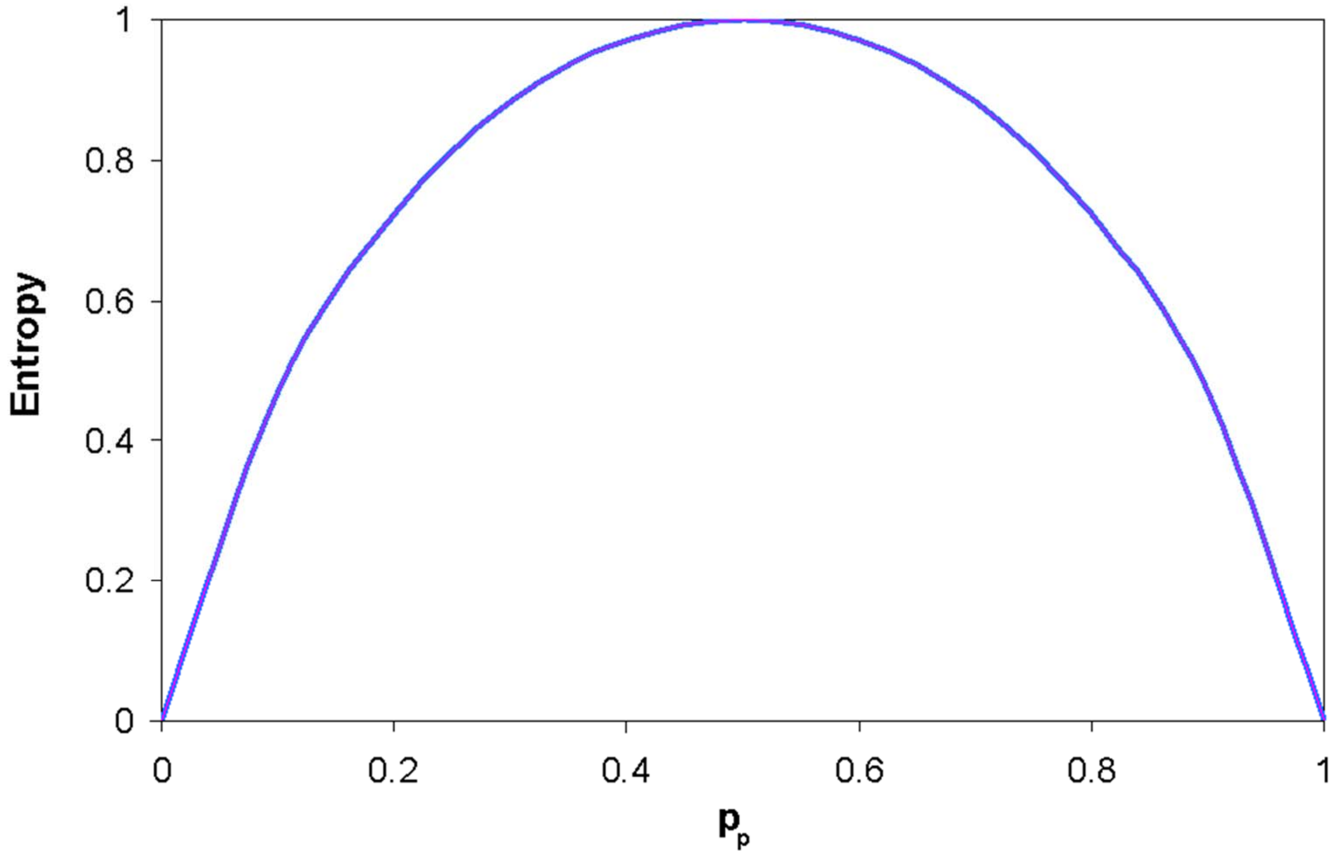
From Spikes: Exploring the Neural Code,
by Reike et al, Bradford 1999.

…and H.

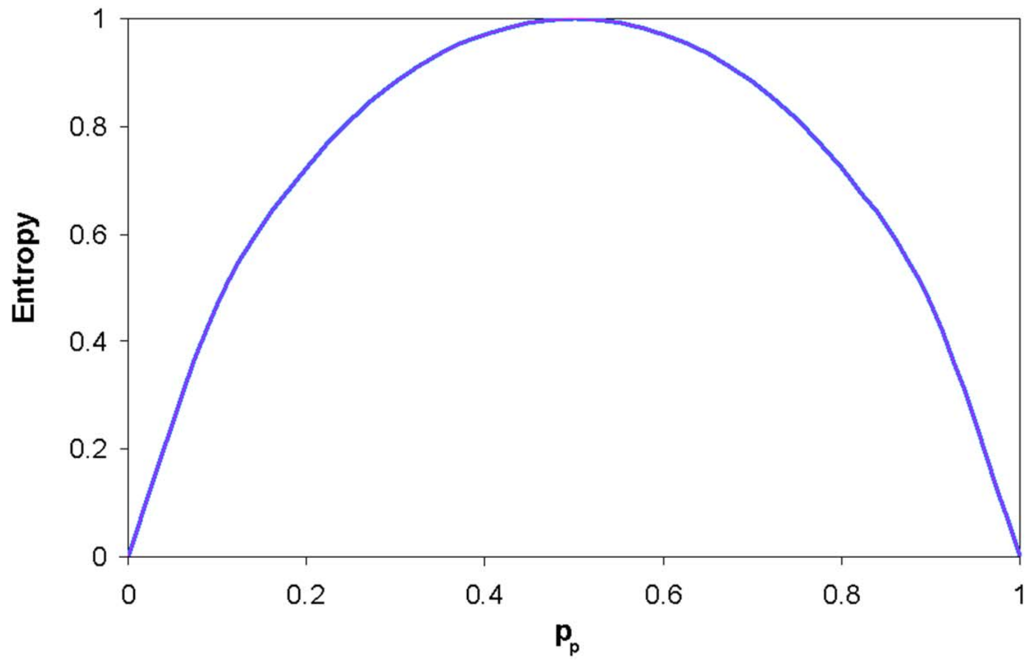
In Spikes they also eliminate H (e.g., as the Hamiltonian). But, others, I, along with Shannon, prefer to honor Hartley. Thus, H for entropy . . .

From: Tom Carter’s *Lecture Notes on Information Theory and Entropy,*
Prepared for Complex Systems Summer School, Santa Fe, June 2012.
sustan.csustan.edu/~tom/

Entropy (2 outcomes)



Entropy: Properties



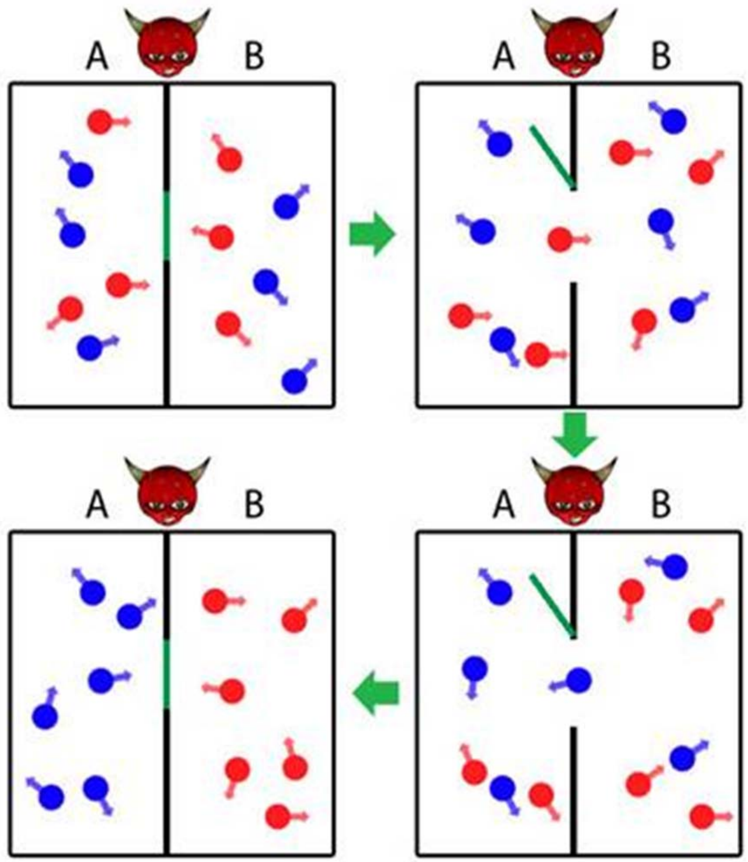
$$0 \leq H(P_n) \leq \log(n)$$

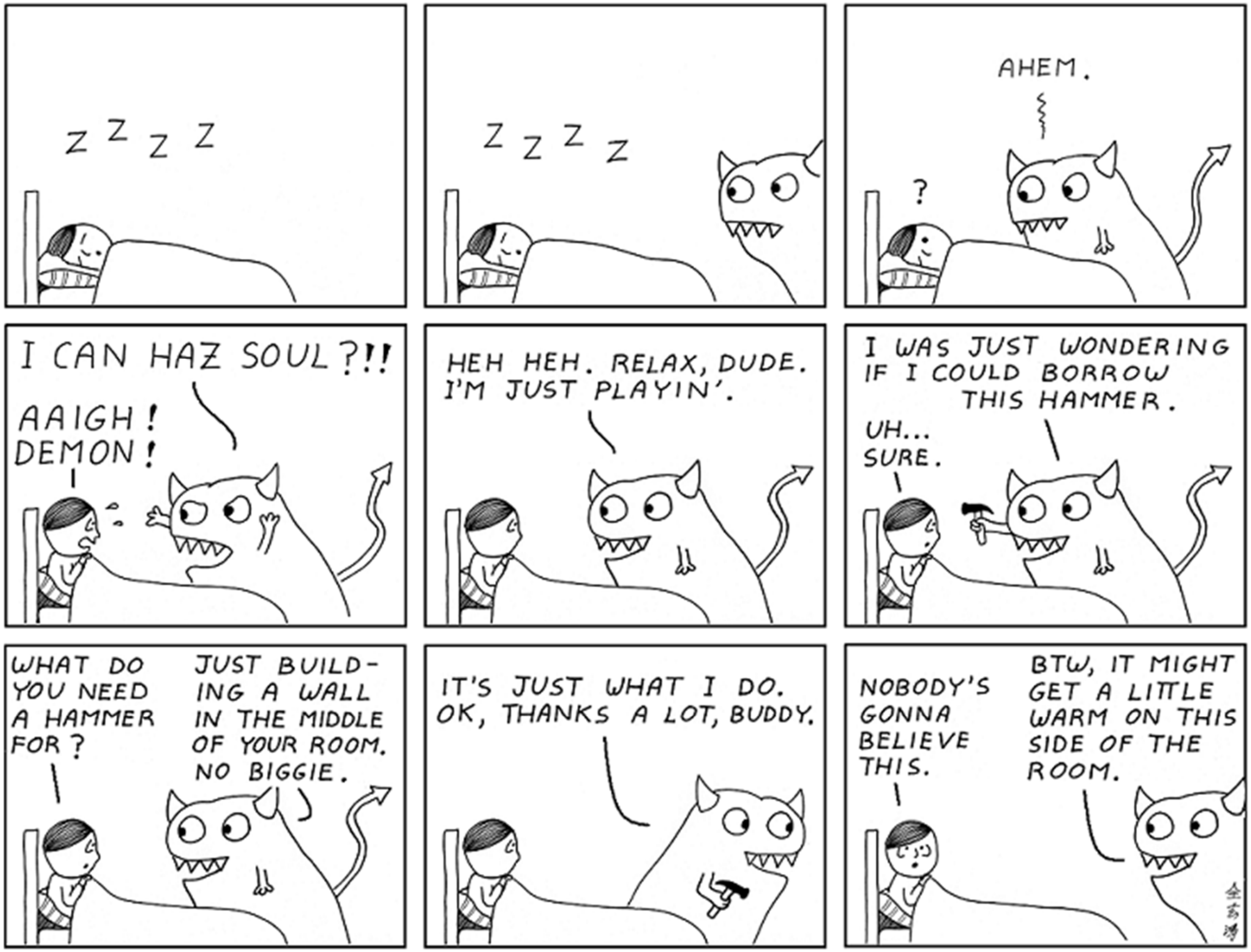
Entropy is maximized if P is uniform.

Entropy: In Thermodynamics

- 19th century Physics…
- Matter and energy
- *Entropy* is the heat loss incurred during work
- Two laws of thermodynamics:
  - First Law: Energy is conserved.
  - Second Law: Entropy always increases until it reaches a peak.

Maxwell's Demon





Maxwell's Demon

only the least scariest demon... EVAH

From: http://www.lastwordonnothing.com/2010/11/23/abstruse-goose-maxwells-demon/

Maxwell's Demon…

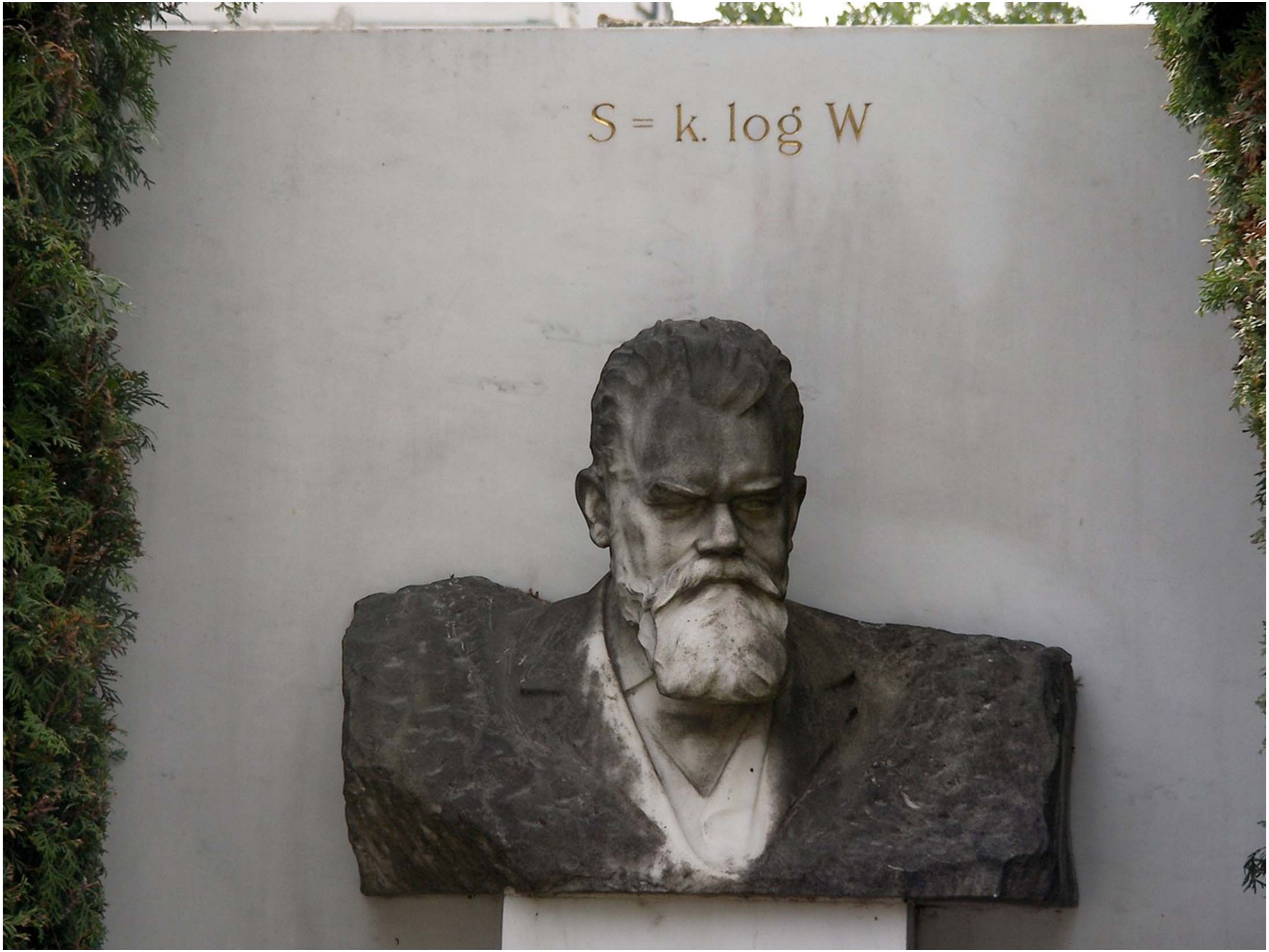
- Szillard (1929): Demon acquires "information" when deciding to open the door. Thus entropy of the entire system is conserved, preserving the 2nd Law.

- Landauer (1960s): It is not the act of measurement, but the act of erasing memory that increases the entropy.

Boltzmann: Statistical Mechanics

- Large scale properties emerge from microscopic properties (*macrostates* & *microstates*).
- *Statistical* approach predicts the average behavior of large ensembles of molecules: bridges classical mechanics with thermodynamics.
- An isolated system will more likely be in a more probable macrostate than in a less probable one.
- Boltzmann entropy is a function of the number of microstates that could give rise to a macrostate.

$$S = k \log(W)$$


S = k. log W

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References

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- Tom Carter, *Lecture Notes on Information Theory and Entropy*. Prepared for Complex Systems Summer School, Santa Fe, June 2012.

Assignment

- Process a large English text (an entire book, e.g.) and compute the top ten:
  - Most frequent letters (a/A..z/Z)
  - Most frequent first letters in words
  - Most frequent last letters
  - Other ideas (bigrams, trigrams, doubles, letters that follow ‘e’, 2-letter words, 3-letter words, 4-letter words)
- Most frequent letters in texts in other languages (French, German, Italian, Spanish)