Computer Graphics

Curves

Based on slides by Dianna Xu, Bryn Mawr College

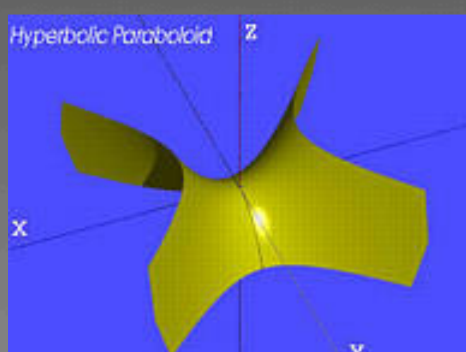
The World is not Flat

- A way to represent true mathematical curvature.
- Rendering will convert the model back to many flat polygons.
- How closely the flat polygons approximate the original model can be controlled.

Simple Curves and Surfaces

- Lines and planes
- Conics

- Quadric surfaces



Ellipsoid
Hyperbolic paraboloid
Hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Mathematical Representations of Curves and Surfaces

- A system of (polynomial) equations
- Parametric form

$$x = \frac{2t}{1+t^2} \qquad y = \frac{1-t^2}{1+t^2}$$

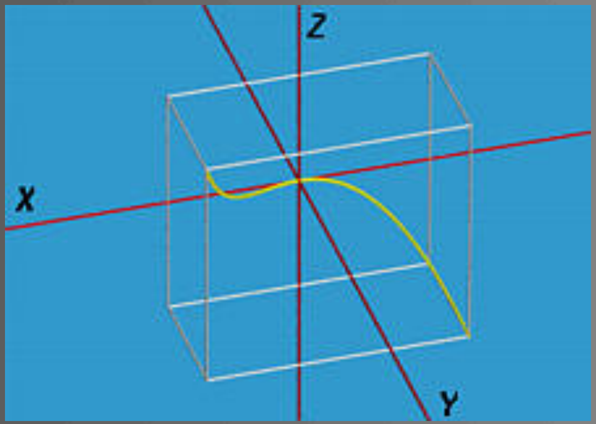
- Implicit form

$$x^2 + y^2 = 1$$

Parametric Curves

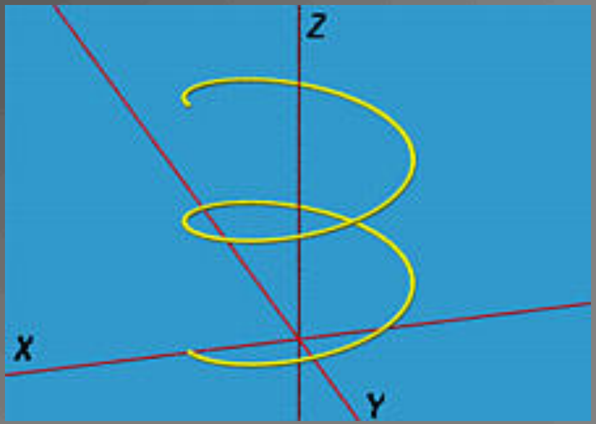
- Cubic

$$f(u) = (u, u^2, u^3)$$



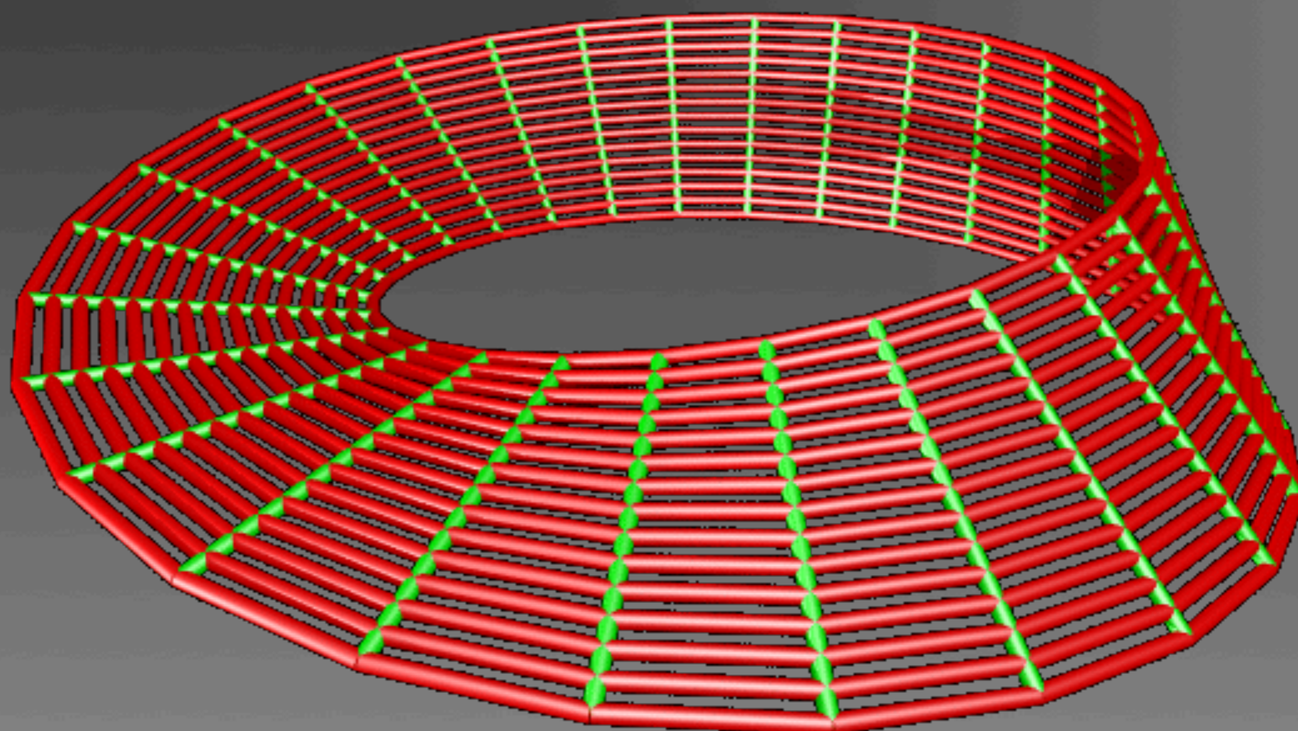
- Helix

$$f(u) = (a\cos(u), a\sin(u), bu)$$

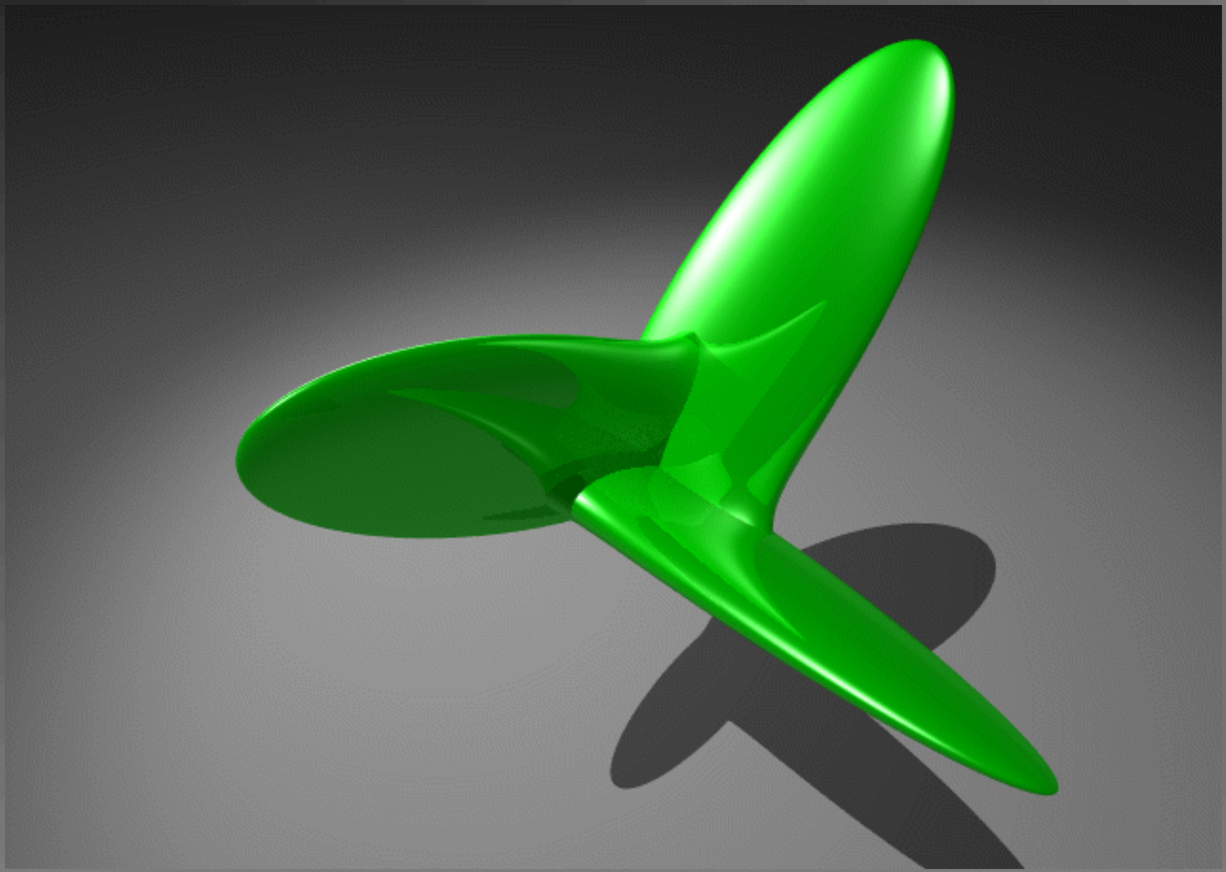


Möbius Band

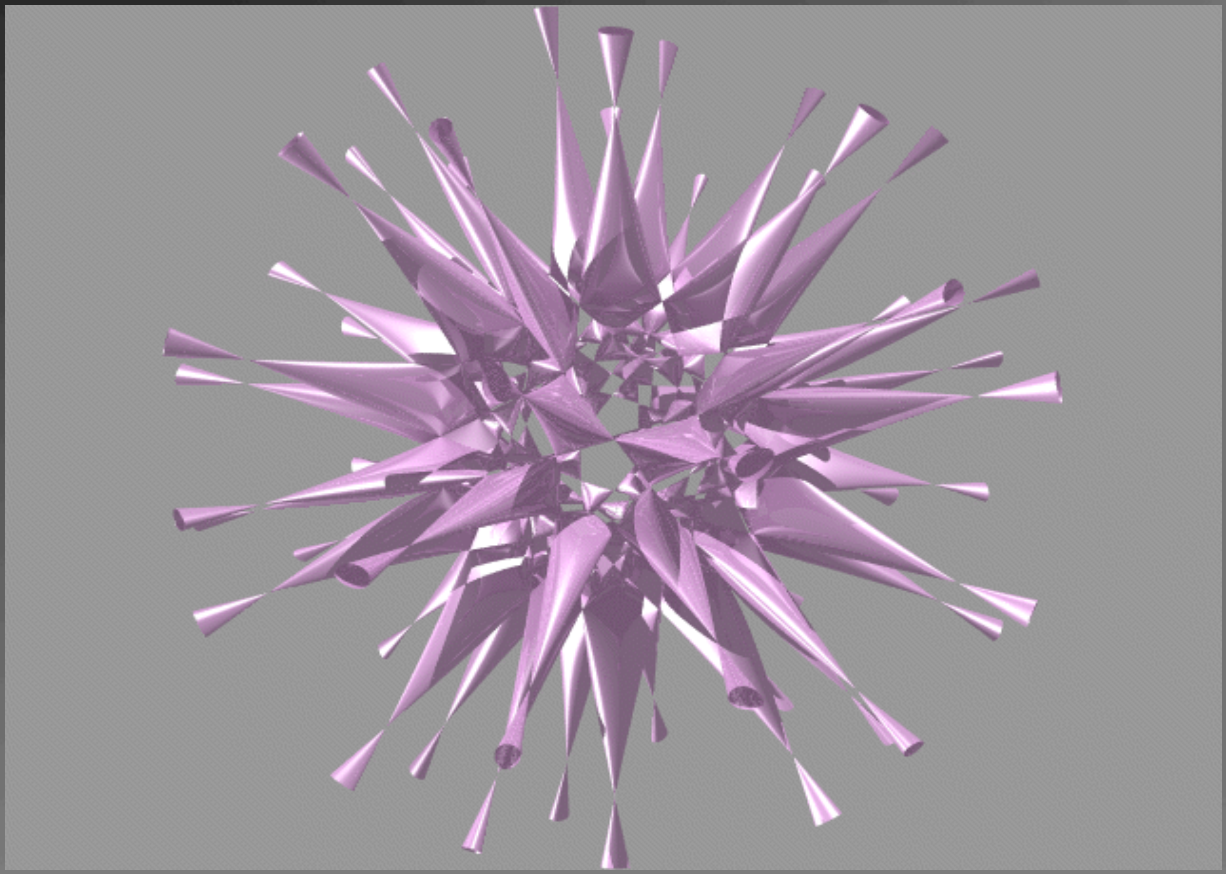
$$f(u,v) = \begin{matrix} \cos(u) + v\cos(u/2)\cos(u) \\ \sin(u) + v\cos(u/2)\sin(u) \\ v\sin(u/2) \end{matrix}$$



Boy's Surface



Barth's Dectic



Cubic Curves

- Parametric form:

$$f(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \sum_{k=0}^{3} c_k u^k$$

- The shape of the curve depends on the coefficients, which are derived from control points.

Cubic Interpolating Curves

- Given 4 control points p0, p1, p2, p3.
- Interpolation means the curve must pass through the above points.
- Solve the system of equations, with u evenly spaced, i.e. u=0, 1/3, 2/3, 1

Cubic Interpolating Curves

- The system of equations:

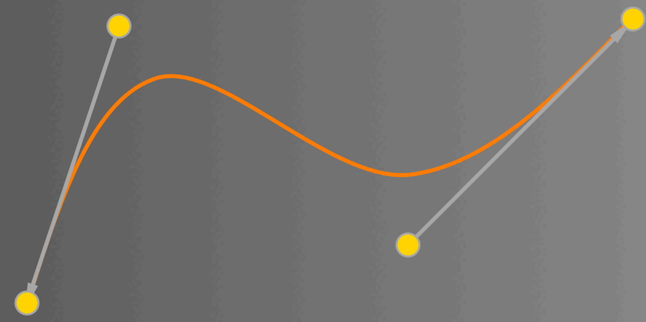
$$p_0 = c_0$$

$$p_1 = c_0 + \frac{1}{3}c_1 + (\frac{1}{3})^2 c_2 + (\frac{1}{3})^3 c_3$$

$$p_2 = c_0 + \frac{2}{3}c_1 + (\frac{2}{3})^2 c_2 + (\frac{2}{3})^3 c_3$$

$$p_3 = c_0 + c_1 + c_2 + c_3$$

Cubic Bézier Curves



- Bézier curves interpolate the endpoints.
- Thus the equations at p0 and p1 are exactly the same as cubic curves
- Bézier proposed to use middle points to approximate derivatives at endpoints.

$$p'(0) = \frac{p_1 - p_0}{1/3} = 3(p_1 - p_0)$$

$$p'(1) = \frac{p_3 - p_2}{1/3} = 3(p_3 - p_2)$$

Bézier Curves

- Given $n+1$ control points, $p_0, p_1 \cdots p_n$ the curve is specified by

$$C(u) = \sum_{i=0}^{n} B_{i,n}(u) p_i$$

- Where $B_{n,i}(u)$ is the Bernstein Polynomial:

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

Bézier Equations

- Linear

$$C(u) = (1-u)p_0 + up_1$$

- Quadratic

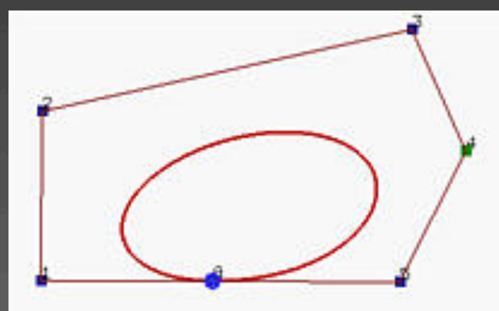
$$C(u) = (1-u)^2 p_0 + 2(1-u)up_1 + u^2 p_2$$

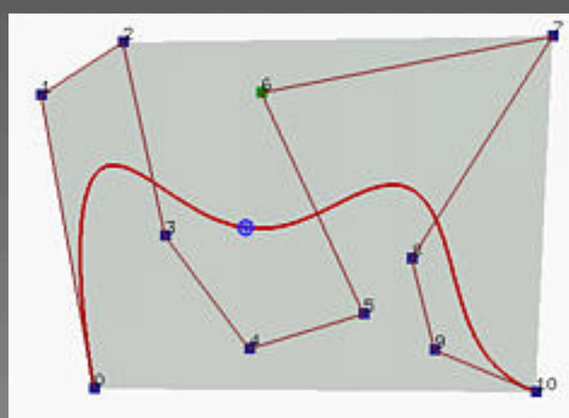
- Cubic

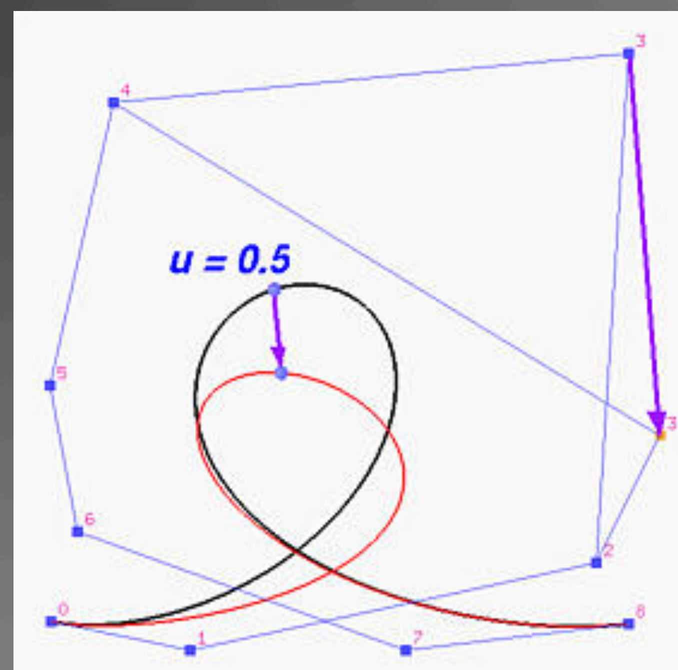
$$C(u) = (1-u)^3 p_0 + 3(1-u)^2 up_1 + 3(1-u)u^2 p_2 + u^3 p_3$$

Bézier Curves: Examples

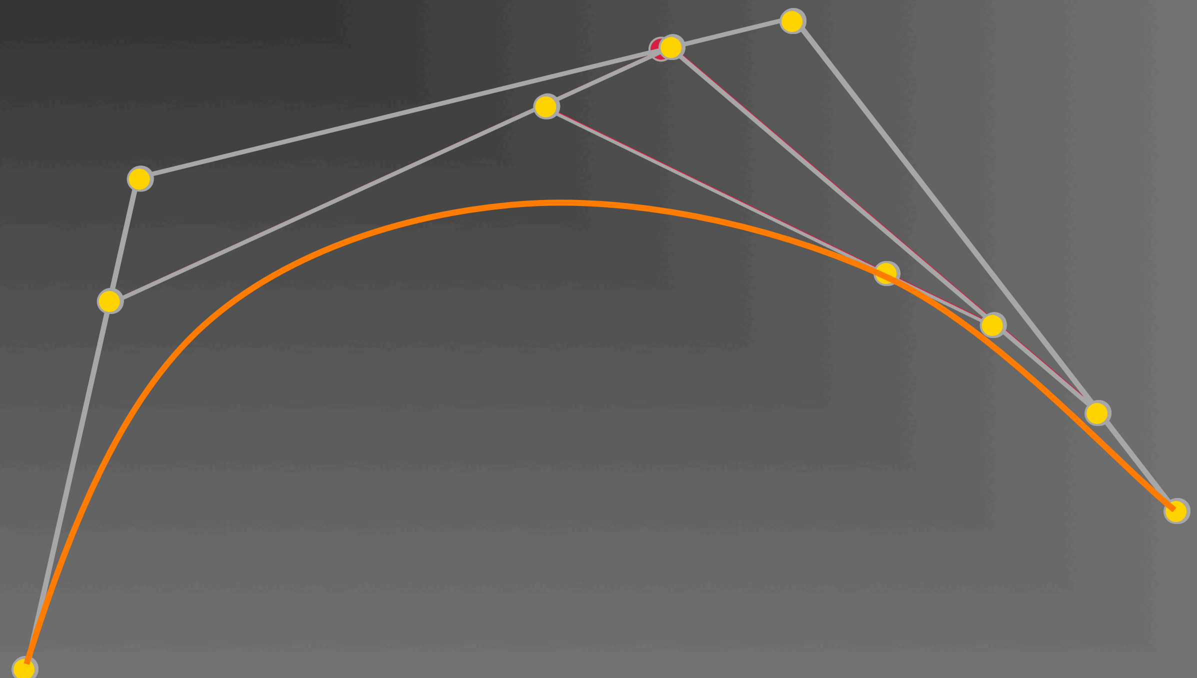
- Changing the position of a control point will change the shape of the curve







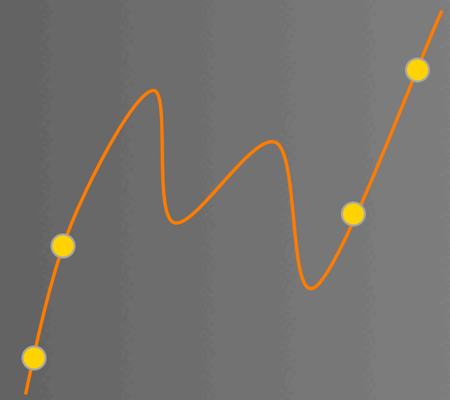
De Casteljau algorithm



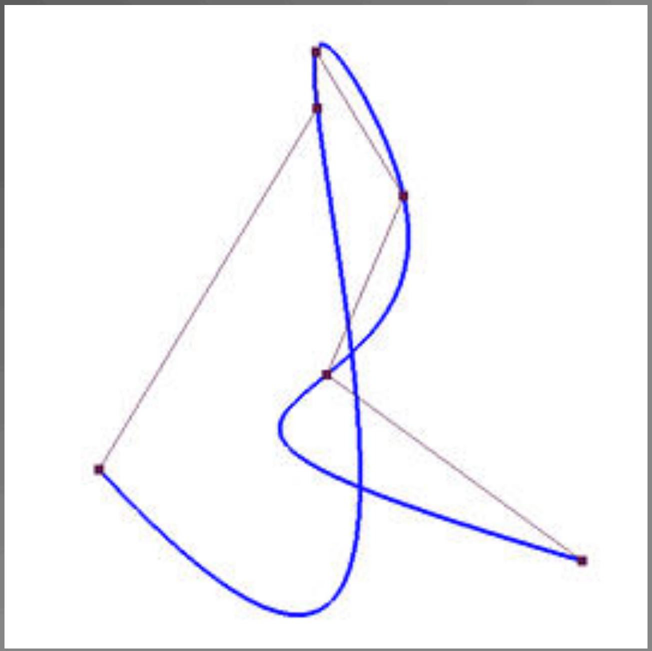
$$u = 0.75$$

Wiggling Effect

- Happens when interpolating, forcing curve through data points



- Wiggling gets much worse with higher degree

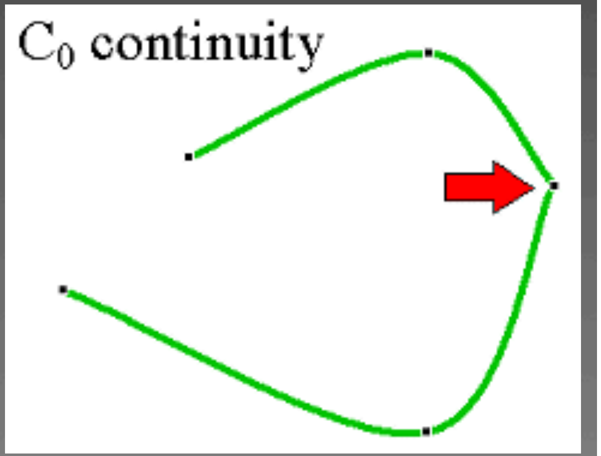


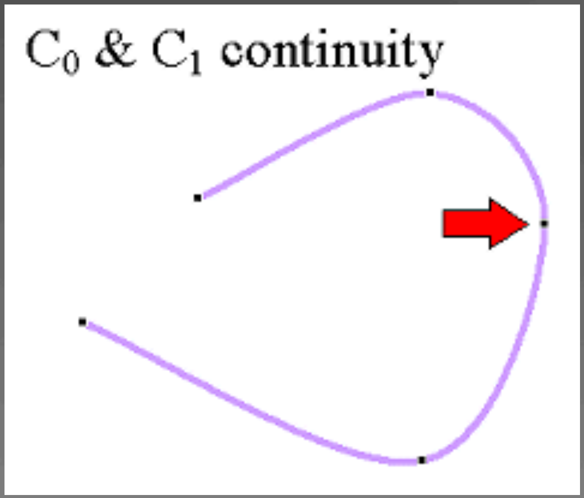
Splines

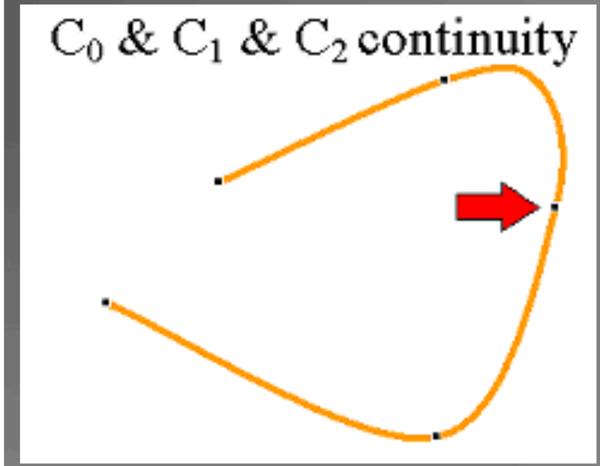
- Use SEVERAL polynomials
- Complete curve consists of many pieces
- All pieces are of low order
  - Third order (cubic) is the most common
- Pieces join smoothly as measured by continuity.

Parametric Continuity

- C^0 – common endpoint
- C^1 – tangent vectors also agree
- C^2 – change in tangents also agree
- C^k – 0th through kth derivatives match







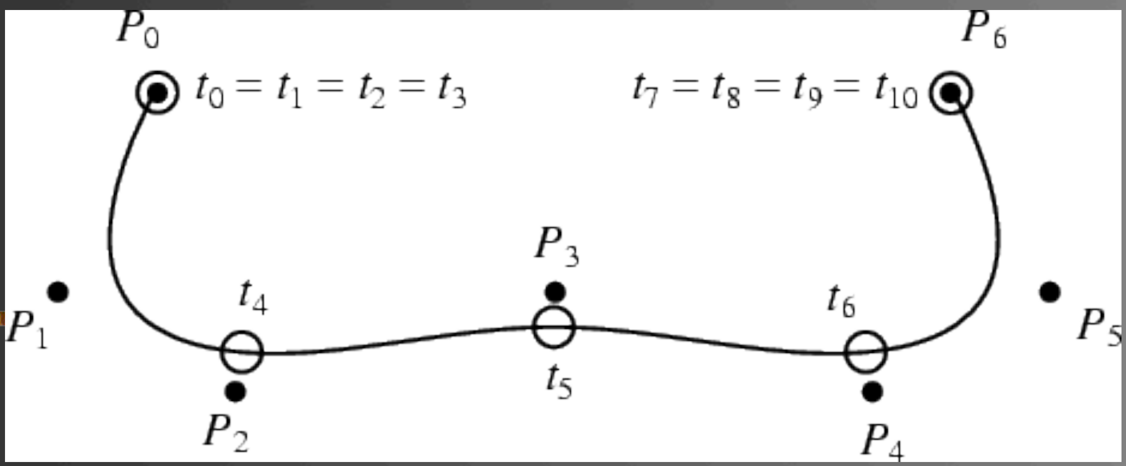
Joining Bézier Segments

- Joining Bézier segments requires additional constraints at endpoints to match up derivatives in order to satisfy continuity conditions.
- Composite Bézier curves can achieve C2 continuity.
- Addition systems of equations much be solved.

B-splines

- B-splines are similar to Bézier curves, just with a different basis (blending) function.
- The new basis function provides:
  - Built-in continuity at joint points – the basis functions themselves are C2 continuous.
  - Local support – control points only influence local section of the curve.

Knots



- The sequence of values is known as a knot vector $t_0 \le t_1 \le \ldots \le t_m$
- The knots specify the joining points of the segments, much as knots joining strings.
- Knots may be equal.
- If there are m+1 knots and n+1 control points, the degree of the curve is p=m-n-1
- When internal knots are evenly spaced, the resulting B-spline is called uniform.

B-Spline Formulation

- Equation

$$S(t) = \sum_{i=0}^{n} N_{i,p}(t)\,p_i, t_0 \le t \le t_m$$

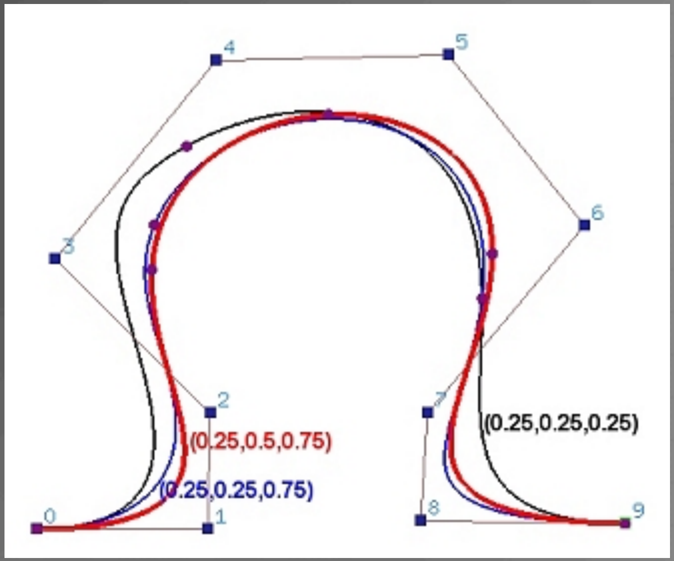
- where

$$N_{i,0}(t) = \begin{cases} 1, t_i \le t \le t_{i+1} \\ 0, otherwise \end{cases}$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t)$$

Modifying the Knots

- Three curves defined by 10 (n=9) control points and is of degree 6.
- Their internal knot vectors are
  - (0.25,0.5,0.75) – red curve,
  - (0.25,0.25,0.75) – blue curve
  - (0.25,0.25,0.25) – black curve.



Other Spline Curves

- Rational B-splines
  - Basis function is rational
- NURBs
  - Non-Uniform Rational B-splines