Computer Graphics

Geometry

Based on Slides by Dianna Xu, Bryn Mawr College

Coordinate-Free Geometry

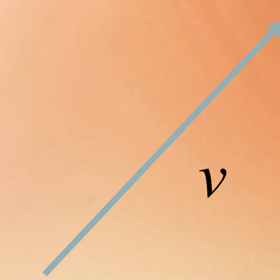
- Most geometric results are independent of the coordinate system
- The product of a matrix and a vector can have many meanings
  - A change of coordinate system
  - A transformation of space
  - A projection
- Basics of affine geometry
  - Scalars, vectors and points

Scalars

- Members of sets which can be combined by two operations
  - Addition
  - Multiplication
  - Obeying some fundamental axioms (associativity, commutivity, inverses)
- Scalars alone have no geometric properties

Vectors

- **Physical definition: a vector is a quantity with two attributes**
  - **Direction**
  - **Magnitude**
- **Examples include**
  - **Force**
  - **Velocity**
  - **Directed line segments**
    - **Most important example for graphics**
    - **Can map to other types**



Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom

Conventions

- Points will be upper-case Roman letters, e.g. P, Q and R
- Vectors will be lower-case roman letters, e.g. u, v and w
- Scalars will be represented as
  - lower case Greek letters, e.g. α, β and γ
  - And a, b and c in programs

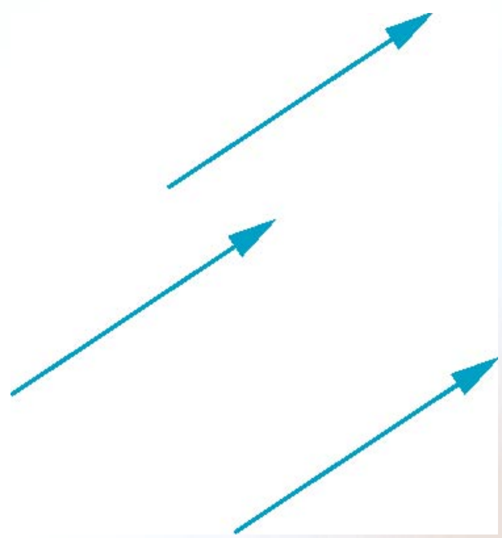
Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u = \alpha v$
  - Vector-vector addition $w = u + v$
- Vector space expressions

$$u = v + 2w - 3r$$

Vectors Lack Position

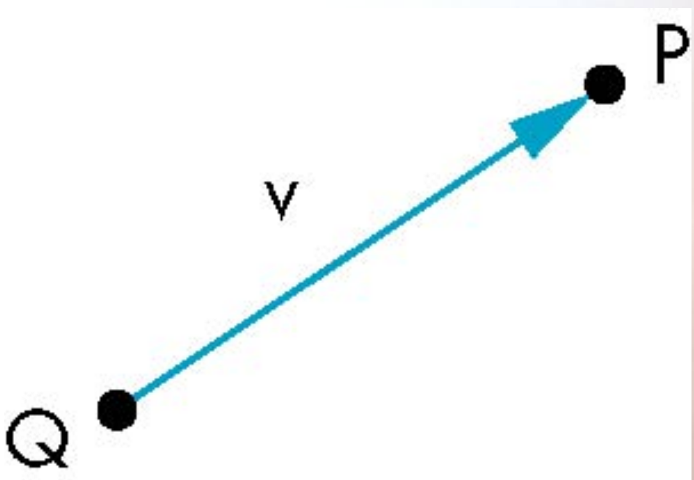
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- **Location in space**
- **Operations allowed between points and vectors**
  - **Point-point subtraction yields a vector**
  - **Equivalent to point-vector addition**



$$P = v + Q$$

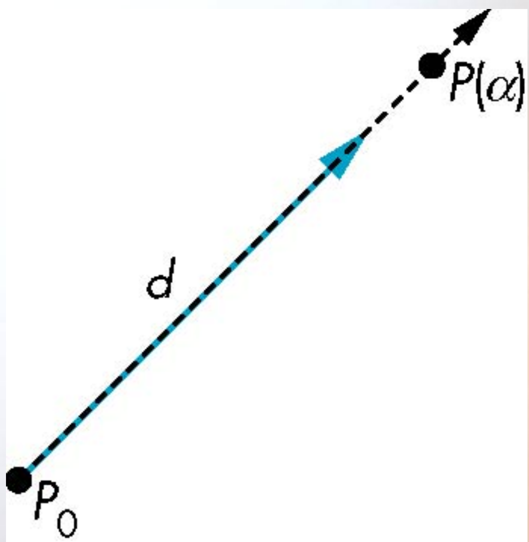
$$v = P - Q$$

Affine Spaces

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $\vec{1} \bullet P = P$
  - $\vec{0} \bullet P = P$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha) = P_0 + \alpha d$
  - Set of all points that pass through P_0 in the direction of the vector d



Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = \alpha x_0 + (1 - \alpha) x_1$$

$$y(\alpha) = \alpha y_0 + (1 - \alpha) y_1$$

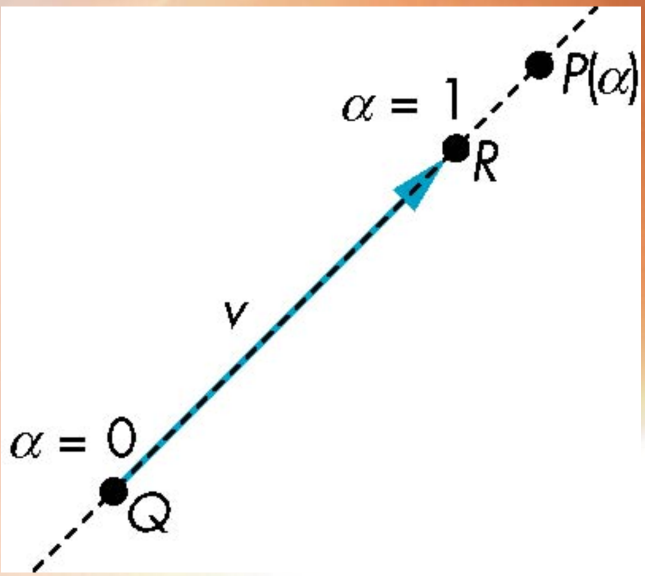
Line Segments and Affine Combinations

- **If α >= 0, then $\mathrm{P}(\alpha)$ is the *vector* leaving P_0 in the direction d**

  If we use two points to define v, then

$$P(\alpha) = Q + \alpha(R - Q) = Q + \alpha v = \alpha R + (1 - \alpha)Q$$

For $0 <= \alpha <= 1$ we get all the points on the *line segment* joining R and Q



Affine Combinations

- Consider the "sum"

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$$

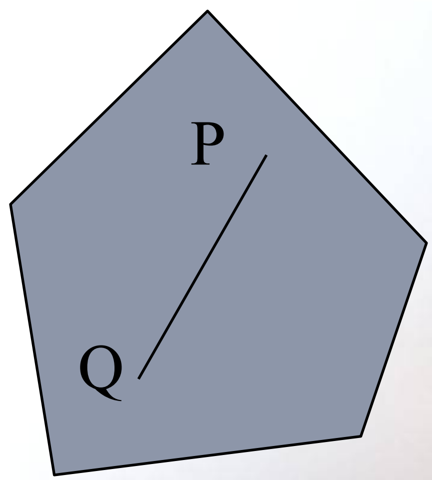
$$\forall \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$$

- We have the *affine combination* of the points $P_1, P_2, \ldots .. P_n$

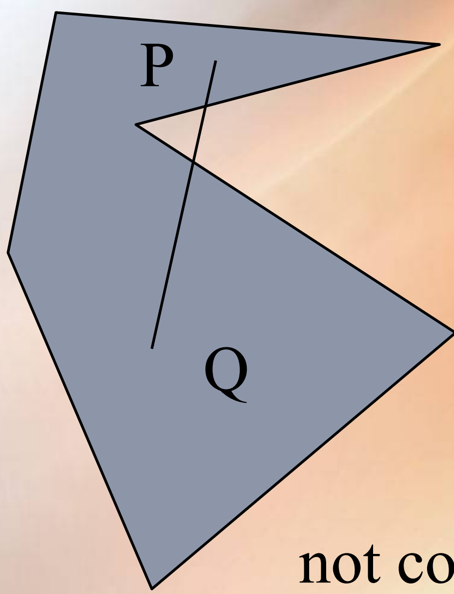
- If, in addition, $\alpha_i >= 0$, we have the *convex combination* of $P_1, P_2, \ldots .. P_n$

Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



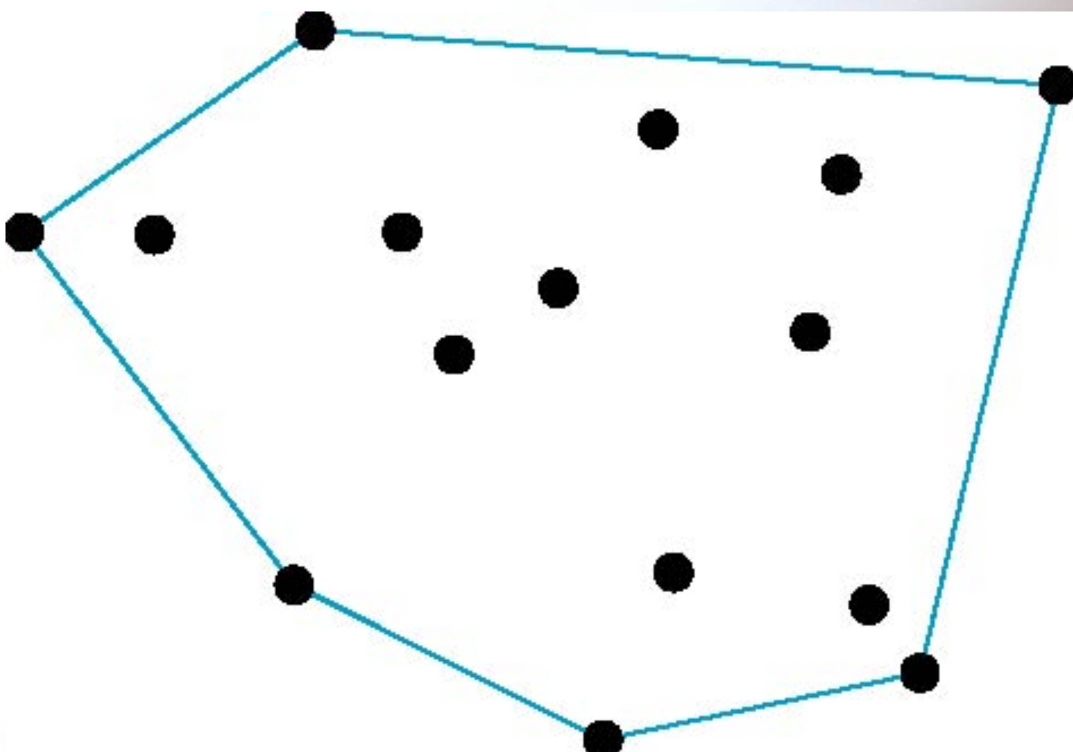
convex



not convex

Convex Hull

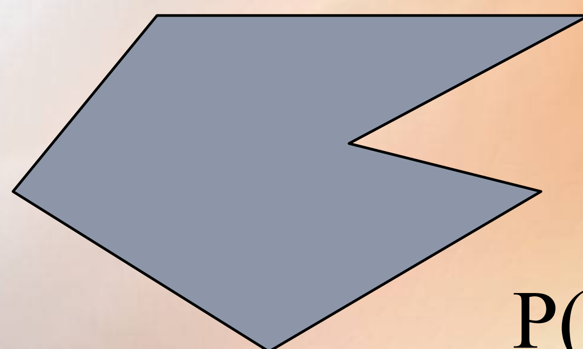
- Smallest convex object containing $P_1, P_2, \ldots\ldots P_n$
- Formed by "shrink wrapping" points



Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
  - Linear functions give planes and polygons

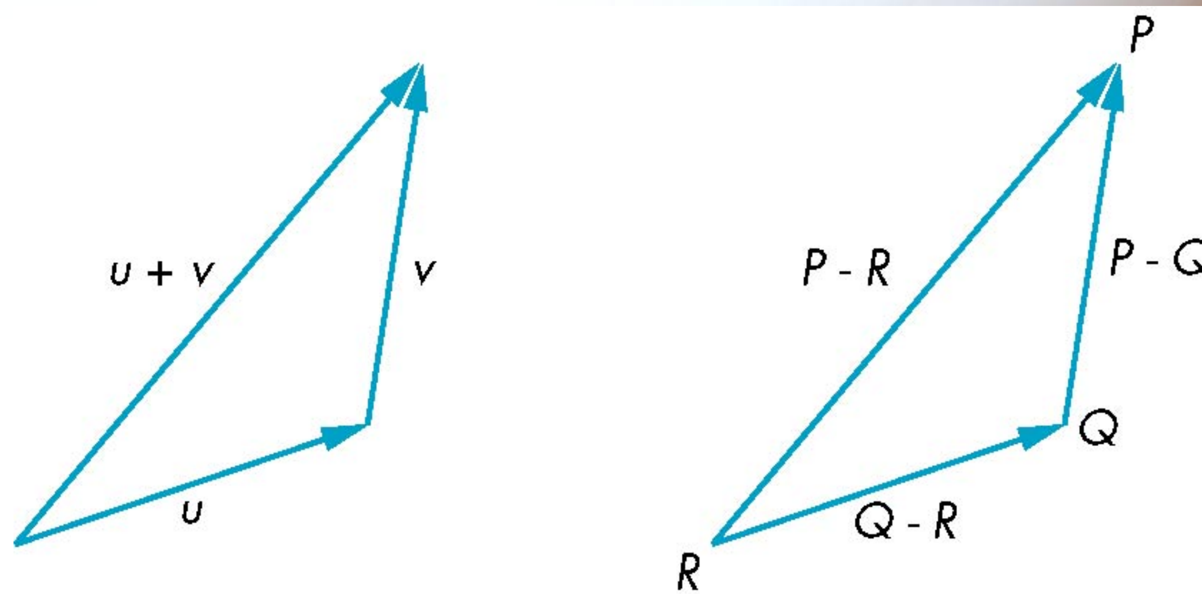
$P(\alpha)$



$P(\alpha, \beta)$

Planes

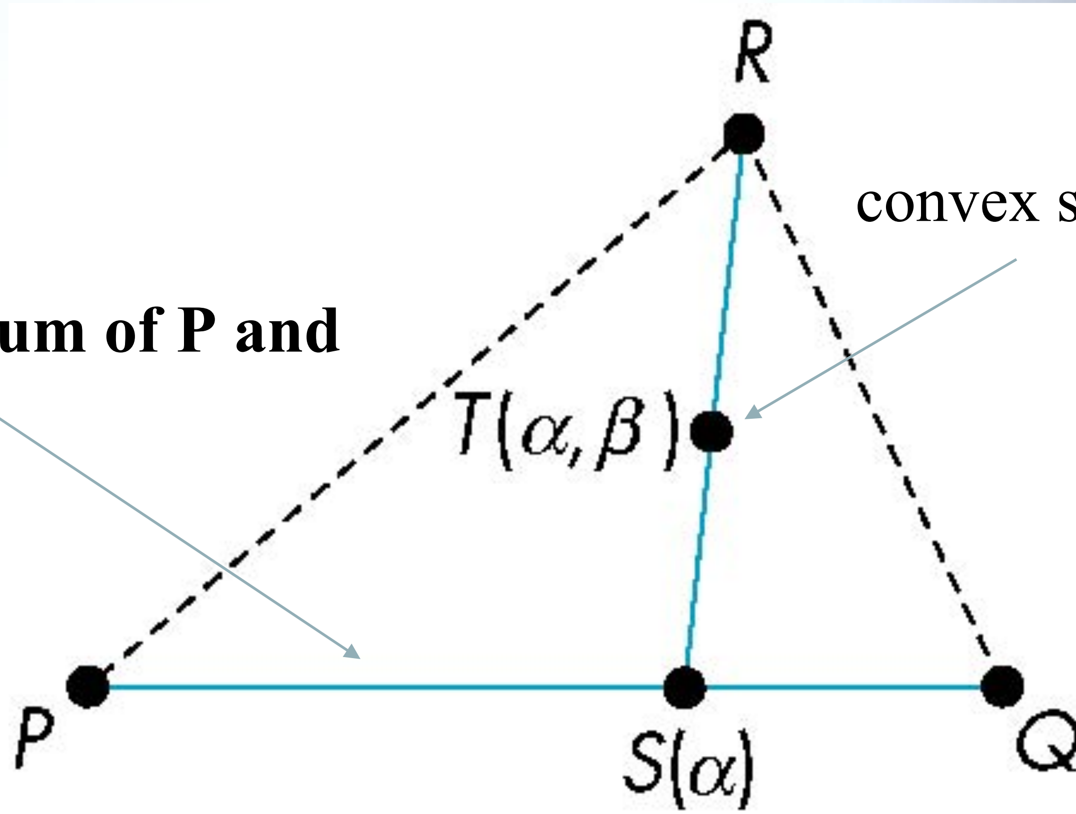
- **A plane be determined by a point and two vectors or by three points**



$$P(\alpha, \beta) = R + \alpha u + \beta v$$

$$P(\alpha, \beta) = R + \alpha(Q - R) + \beta(P - Q)$$

Triangles



convex sum of S(α) and R

convex sum of P and Q

for $0 <= \alpha, \beta <= 1$, we get all points in triangle

Euclidean Geometry

- Euclidean geometry is an extension of affine geometry with one additional operation
- Inner product (dot product)
  - Maps two vectors to a scalar

$$\vec{u} \bullet \vec{v} = \sum_{i=0}^{d-1} u_i v_i$$

- Length

$$\left|\vec{v}\right| = \sqrt{\vec{v} \bullet \vec{v}} \Leftrightarrow \vec{v} \bullet \vec{v} = \left|\vec{v}\right|^2$$

Normalization

- A vector having length of exactly 1 is called the *unit vector*.
- A vector of non-zero length can be ***normalized*** by dividing its own length

$$\hat{v} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

- Distance btw two points

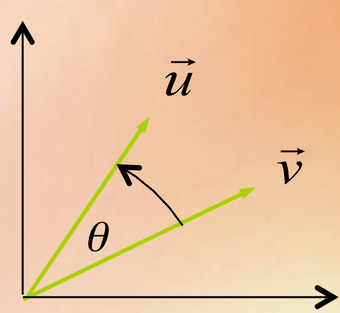
$$dist(P,Q) = \left|P - Q\right|$$

The Dot Product

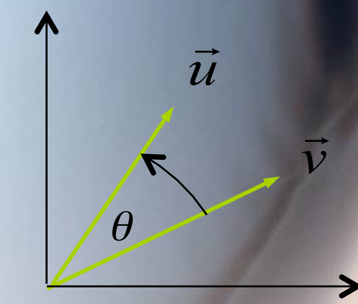
$$\vec{u} \bullet \vec{v} = u_0 v_0 + u_1 v_1 + \ldots + u_{d-1} v_{d-1} = \sum_{i=0}^{d-1} u_i v_i$$

- The dot product of two vectors measures the difference between the directions in which the two vectors point

$$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta$$



Geometric Significance of the Dot Product

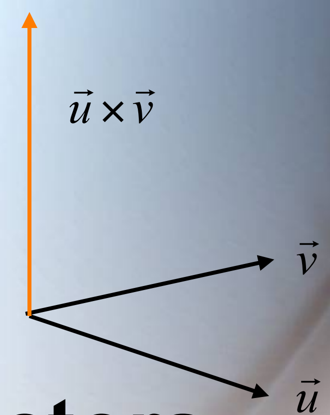


- **Angle btw two vectors**

$$ang(\vec{u},\vec{v}) = \cos^{-1}\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\hat{v}\bullet\hat{u}\right)$$

- **If two vectors are perpendicular, their dot product is 0**
- **If two vectors point in the same half plane, their dot product is non-negative**

The Cross Product



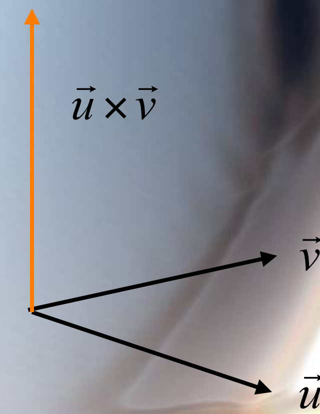
- **The cross product of two vectors returns a new vector perpendicular to both**

$$\vec{u}\times\vec{v} =< u_y v_z - u_z v_y , u_z v_x - u_x v_z , u_x v_y - u_y v_x >$$

- **Also given by the pseudo-determinant**

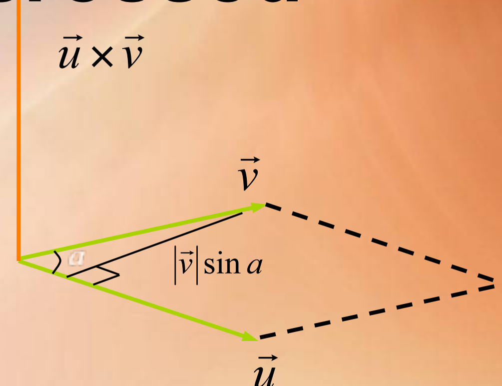
$$\vec{u}\times\vec{v} = \begin{bmatrix} i & j & k \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = i(u_y v_z - u_z v_y) - j(u_x v_z - u_z v_x) + k(u_x v_y - u_y v_x)$$

Geometric Significance of the Cross Product



- The length of the cross product is the signed area of the parallelogram formed by the two vectors crossed

$$\left|\vec{u}\times\vec{v}\right| = \left|\vec{u}\right|\left|\vec{v}\right|\sin a$$



- The direction of the cross product is given by the right hand rule

Normals

- **Every plane has a vector n normal (perpendicular, orthogonal) to it**
- **From point-two vector form** $P(\alpha,\beta) = P_0 + \alpha u + \beta v$ **we know we can use the cross product to find n** $=$ **u** $\times$ **v and the equivalent form**

$$(P(\alpha,\beta) - P_0) \bullet n = 0$$