FIRST EXPERIENCES WITH WRITING PROOFS

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"Organized writing and organized thinking are intertwined."

Writing is a significant and essential part of higher mathematics. Unlike many of your previous mathematics classes, this course has a greater emphasis on concepts and abstract ideas. The primary goal of this course is to teach you how to prove mathematical statements. This consists of two parts: Reasoning to determine why a statement is true, and then explaining this reasoning to others.

There is a good chance that you have not done much writing in a math class before. Consequently you might be wondering why writing is required in your math class now. The reason is that mathematics, particularly advanced mathematics, is about ideas. In math classes beyond the introductory level, the ideas encountered are more complex and sophisticated and include concepts which cannot be expressed using simple computations and formulas. Thus, being able to write clearly is important for expressing these ideas, and writing is as important a mathematical skill as being able to solve equations. Mastering the ability to write clear mathematical explanations is important for non-mathematicians as well. As you continue taking math courses in college, you will come to know more mathematics than most other people. When you use your mathematical knowledge in the future, you may be required to explain your thinking process to another person (e.g., your boss, a co-worker, or an elected official), and it will be quite likely that this other person will know less math than you do. Learning how to communicate mathematical ideas clearly can help you advance in your career.

You will also find that writing good mathematical explanations will improve your knowledge and understanding of the mathematical ideas you encounter. Putting an idea on paper requires careful thought and attention. As a result, mathematics which is written clearly and carefully is more likely to be correct. The process of writing will help you to learn and retain the concepts which you will be exploring in this class as well as in other courses.

When writing mathematics, there are certain conventions that should be followed. In informal communication (such as at the board, in one-on-one verbal communication, or when taking notes for your own use) it is common to use abbreviated notation, write sentence fragments, or engage in other standard sloppiness. However, when writing up formal proofs there are stricter guidelines that should be adhered to. More care must be taken in explanations and exposition, and the English language should be used effectively. Writing mathematics well is a skill that takes time and practice to learn, and the following list is meant to provide the beginner with aspects of style to consider, conventions to be aware of, and common pitfalls to avoid.

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What is a Proof?

A proof is a demonstration that a certain statement or proposition is true. It is nothing more than an argument, written in the language of the writer (in our case, English), that presents a line of reasoning explaining why the statement follows from known facts. Although a proof may contain symbols, equations, or calculations as the author tries to present a line of reasoning, it is first and foremost an argument. Consequently most proofs contain many more words than symbols.

A mathematical proof is judged on the degree to which it is:

(1) Correct
(2) Clear
(3) Concise

First, the reasoning in a mathematical proof must be correct. No untrue statements should be made, definitions should be used properly and precisely, and the rules of logic should be followed.

Second, the explanation contained in a mathematical proof should be clear. When you write up a proof of a statement, you should consider it to be an argument as to why that statement is true. After writing a proof, you should consider the following: Suppose that there was person familiar with the definitions and notation used in our course, and suppose this person was capable of reasoning logically. If this person did not initially see why the proposition you are considering is true, would they be able to read and understand your proof, and be convinced of the statement's truth after they had finished reading it? If the answer is "no", then your proof is insufficient.

Third, your proof should be concise. Write as simply and directly as possible. Avoid the use of ponderous or pretentious prose, and remove any unnecessary words or phrases. It is surprising how often one can take a piece of writing and improve it merely by removing portions. This does not mean that your final product must be short, or that you must leave out details. Instead, you should write so that every word, phrase, and sentence contributes to what you are trying to communicate. For example, there is no need to say: "and now we prove the statement". Simply prove it. Likewise, there is no reason to remark: "we have proven the claim" at the end of a proof, since the symbol $\square$ says precisely that already. Do not repeat yourself in your writing, and do not use superfluous phrases. When you proofread a paper for the first time, ask yourself after every sentence whether the reader would be any less informed you removed it. If the answer is "no", you should take it out. In short, you should follow the advice of William Strunk and "make every word tell".

> *"Vigorous writing is concise. A sentence should contain no unnecessary words, a paragraph no unnecessary sentences, for the same reason that a drawing should have no unnecessary lines and a machine no unnecessary parts. This requires not that the writer make all sentences short or avoid all detail and treat only subjects in outline, but that every word tell."*
>
> — *William Strunk*

Furthermore, keep in mind that writing concisely takes far more time and effort than writing at length — it is for this reason that Pascal once at the end of a long letter apologized for not having had the time to write a short one. Despite this, writing in a concise and succinct manner is well worth the effort because it contributes to producing work that is clear, well organized, and direct.

Common Proof Techniques

A proof is an argument, and it is difficult to describe a general method for creating an argument that will work in any situation. Often, coming up with a proof will involve some ingenuity or creativity, and there is no algorithm that can tell you how to prove every statement that you come across.

Nonetheless, the there are certain techniques that are often useful when writing proofs. A few of the more common ones are described here.

Direct Proof: One establishes the conclusion by logically combining the axioms, definitions, and earlier theorems

Proof by Contradiction: (also known as Reductio Ad Absurdum) One shows that if some statement were false, a logical contradiction occurs, and hence the statement must be true.

Proof by Contrapositive: Rather than proving the statement "If P, then Q", one proves the contrapositive "If not Q, then not P". Although the contrapositive is logically equivalent to the original statement, sometimes the form of the contrapositive makes it easier to work with. (Also, many proofs by contradiction can often be rewritten as proofs by contrapositive, and in this case the proof by contrapositive is usually clearer and more succinct.)

Proof by Construction: One constructs a concrete example with a given property to show that something having that property exists.

Proof by Exhaustion: (also known as Proof by Cases) One establishes a conclusion by dividing it into a finite number of cases and proving each one separately

Proof by Induction: One proves a base case, and then an induction rule is used to prove an (often infinite) series of other cases

Mechanics

When composing a proof there are several practical issues related to writing that one should consider.

1. Write multiple drafts. Rarely (if ever) does one read a statement, and then immediately write out the final version of a proof. Instead, you read the statement to make sure you understand it. Then you do scratch work until you are convinced you know why the statement is true and how a proof should go. Next, you write out a first draft of the proof. As you do this, you may find a problem and have to start over, or you may realize that a portion you wrote earlier needs to be adjusted. When you finally get a rough draft of your proof that you are satisfied with, you read it through very carefully to make sure it is correct. (Did you use the definitions correctly? Are there any possibilities you forgot to consider? Are the computations done correctly?) Once you are convinced the proof is correct, you read through it again to see if it can be simplified or made clearer. This last step may need to be iterated several times.

2. End your proofs with the symbol $\square$ or Q.E.D. It helps the reader if you indicate when a proof is finished. This is particularly true in a mathematics paper, where one may continue with prose after the end of the proof. The classic way of ending a proof is with the abbreviation Q.E.D., which stands for the latin phrase *quod erat demonstrandum* and which translates as "which was to be demonstrated". In the past 50 years, however, it has become more common to use the symbol $\square$ to indicate the end of a proof. This symbol is sometimes called the "tombstone" or the "Halmos symbol", after the mathematician Paul Halmos who first introduced it. Sometimes the tombstone is also written as a solid black square; $\blacksquare$.

3. Carefully define any symbol or variable that you introduce. If you are told that a divides b, do not simply write $b = ac$ without saying what c is. Instead you should write "$b = ac$, where c is some element of $\mathbb{Z}$". Or, if you are given a rational number x, do not write $x = \frac{p}{q}$ without explaining that p and q are integers and $q \neq 0$. Any time you write a symbol that does not appear in the statement you are proving, you should tell the reader what that symbol stands for.

4. Point out when you use the hypotheses of the statement you are proving. Tell the reader whenever you use a hypothesis to reach a conclusion. If you have the hypothesis that $a, b \in \mathbb{R}$ and $a \neq 0$, and in your proof you have an equation $ab = 0$ and you conclude that $b = 0$, you should explain that you are using the fact that $a \neq 0$. Keep track of the hypotheses and where you use them. If there is a hypothesis in the statement that you do not use, odds are your proof is incorrect.

5. Use the first person plural when writing proofs. It is a convention in the mathematical community to use the first person plural, or "we", when writing papers. This choice has many advantages. It conveys the active and participatory nature of the project, making readers feel involved as they work through the paper. In addition, it is what most people are used to hearing in mathematics classes or talks, and therefore has a familiar cadence which is less likely to cause distraction. Furthermore, it avoids many of the problems encountered with other choices, such as the pretension of the first person singular "I", or the awkward sentences that arise with the third person singular "one".

6. Be very careful when using pronouns, particularly the words "this" and "it". A requirement of good writing is to make it clear to the reader at all times what is the subject under discussion. Use a pronoun only when it is clear what the pronoun is referring to. Phrases such as "This is a consequence of Theorem 2" should be used with caution as they often force the reader to guess what "this" is referring to. The word "it" can also be ambiguous, and has committed countless crimes in mathematics. For example, the connection between extreme points of a function f and roots of f' has sometimes been put as vaguely as this:

It is maximum or minimum when it is a zero of its derivative.

This sentence contains too many pronouns, and it is far from clear what each one refers to. A clearer exposition is:

> When f has a maximum or minimum value at a point x, then x is a root of the derivative function f'.

7. Take care with word order. Because mathematical language conveys complex and precise thoughts, word order can be crucial. Consider these two statements:

> For every $\epsilon > 0$ there exists an integer n such that $0 < 1/n < \epsilon$.

and

> There exists an integer n such that for every $\epsilon > 0$ one has $0 < 1/n < \epsilon$.

The first statement is true; it is a version of what is sometimes called the Archimedean principle for real numbers. The second statement is false.

English Usage

1. Apply the standard rules of English usage. Use punctuation, capital letters, proper spelling, and proper grammar. These devices are used in ordinary English to help the reader see clearly what is being said or implied, and they are used in mathematical writing for the same reasons. Be forewarned that a misplaced piece of punctuation or a grammatically incorrect statement can often change or obscure the meaning of a sentence.

2. Punctuate equations and mathematical symbols. Even when a sentence contains mathematical symbols or equations, the rules of the English language still apply. In fact mathematical expression should be thought of as no different than the words they represent, and should be punctuated accordingly. This applies even to displayed equations so that, for example, if a displayed equation is at the end of a sentence it should end with a period.

3. Write in complete sentences. This is covered by the previous two entries, but bears repeating. It is fine to write

Because $x > 3$, we know $x^2 > 9$ and $x^3 > 27$.

But it is usually not good form to write

$$x > 3, \ x^2 > 9, \ x^3 > 27.$$

4. Use Latin abbreviations correctly. The following table summarizes the meanings of some commonly used Latin abbreviations:

Abbreviation	Latin term	English translation
i.e.	id est	that is
e.g.	exempli gratia	for example
cf.	confer	compare
n.b.	nota bene	note well (or just note)
q.v.	quod vide	which see
viz.	videlicet	namely
et al.	et alii	and others

In particular, the abbreviations i.e. and e.g. are often mistakenly interchanged, and cf. is often misused to mean "see" when it actually means "compare". Also note that there is no period after "et" since it is not an abbreviation.

Style

1. Make sure your writing flows. Avoid writing a succession of loose sentences. Particularly when writing proofs, it is easy to become so engrossed in the mathematics that one forgets to pay attention to English style. The result is often a proof that reads " . . . and then . . . and then . . . and then . . .". Try to use a variety of words in proofs, such as "therefore", "consequently", "it follows that", "we see", "hence", or "thus".

2. Do not start a sentence with a variable or symbol. Although it is perhaps technically correct, it is considered bad style to do so. Usually this can be avoided by simply rewording the sentence; e.g., rather than "n points are on the interior" one would write "The interior contains n points". (There are, of course, some exceptions to this rule. In particular, most people would consider it acceptable to start a sentence with a mathematical term that contains a symbol, especially if it is an uppercase symbol. For example, one should feel free to start a sentence with the word C^*-algebra or the word K-theory.)

3. Do not use common blackboard abbreviations in proofs. For example, write "if and only if" rather than "iff" or $\Longleftrightarrow$, and write "without loss of generality" rather than "WLOG". This also applies to symbols such as $\forall$ and $\exists$. Unless one is writing a paper in mathematical logic, one should write out "for all" and "there exists". This does not apply to symbols or notation used to express mathematical ideas; for example, it is fine to write $a|b$ in place of "a divides b" or $x \in S$ rather than "x is in S".

4. Do not use contractions in formal writing. Thus words such as "don't", "can't", "I'm", and "we've" should be written out.

Resources

On mathematical writing:

S. Krantz, *A Primer of Mathematical Writing*, American Mathematical Society, Providence, Rhode Island, 1997.

P. Halmos, *How to Write Mathematics*, American Mathematical Society, Providence, Rhode Island, 1973.

L. Gillman, *Writing Mathematics Well: A Manual for Authors*, The Mathematical Association of America, Washington, D.C., 1987.

N. J. Higham, *Handbook of Writing for the Mathematical Sciences*, SIAM, Philadelphia, Pennsylvania, 1993.

American Mathematical Society, *A Manual for Authors of Mathematical Papers*, 8$^{\text{th}}$ Ed., pamphlet, 20 pp, American Mathematical Society, 1984.

On general writing and English usage:

W. Strunk and E. B. White, *The Elements of Style*, 4$^{\text{th}}$ Ed., Macmillan, New York, 1979.

H. W. Fowler, *A Dictionary of Modern English Usage*, 2$^{\text{nd}}$ Ed., Revised and Edited by Sir Ernest Gowers, Oxford University Press, 1965.