1. Prove that if m is an even integer, then $m + 7$ is odd. Do this proof in three ways: direct proof, proof by contraposition and proof by contradition.

2. Using proof by contradiction, prove that $\forall n \in \mathbb{Z}, 4 \nmid (n^2 + 2)$

3. Using induction, prove that for all integers $n \geq 1, 2^{2n} - 1$ is divisible by 3, i.e. $3 | 2^{2n} - 1$

4. Given sets A, B, and C in the same universe, determine if each of the following statements is true or false. If it is true, then prove it. If it is false, then give a counter example.

   (a) $C \subseteq A \wedge C \subseteq B \rightarrow C \subseteq A \cup B$

   (b) $C \subseteq A \cup B \rightarrow C \subseteq A \wedge C \subseteq B$

   (c) $\overline{A} \cap (A \cup B) = B \setminus A$

5. Prove that give a set S, the cardinality of its power set is $2^{|S|}$.

6. Prove that if $a_1, a_2, ..., a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are needed to compute the product of these n numbers, no matter how parentheses are inserted into their product.

7. For each of the following, give a *recursive* definition. Remember to indicate the initial terms or base:

   (a) $a_n = \sum_{i=0}^{n} i$

   (b) The sequence that generates the terms 3, 6, 12, 24, 48, 96, 192, ...

   (c) The set of non-negative even numbers

   (d) The set of all even numbers

8. Find explicit formulae for the following recursively defined sequences, and prove correctness using induction.

   (a) $a_k = k - a_{k-1}$, $\forall k \geq 1$, $a_0 = 0$.

   (b) $a_k = 2a_{k-2}$, $\forall k \geq 2$, $a_0 = 1, a_1 = 2$.

9. Prove the correctness of the following algorithm:

```
[Pre-condition: i=1 and sum=0]
while(i<=100)
   sum := sum + i
   i := i + 1
end while
[Post-condition: sum = 1 + 2 + ... + 100]
```

   State your loop invariant clearly.

10. Given the following recursive definition of a set S:

    • Basis: $\lambda \in S$

    • Recursive: $x \in S \rightarrow axa \in S$, $a \in \Sigma$, where Σ denotes some alphabet (set of characters).

    Prove using structural induction, that $\forall s \in S, |s|$ is even.