Trails, Paths and Circuits

CS231

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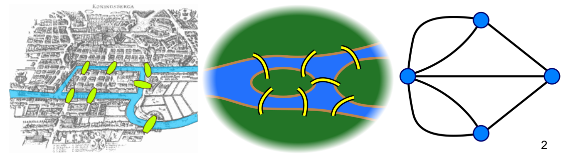
The Seven Bridges of Königsberg

- Leonhard Euler (1736)
- Is it possible to walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?



Walk

- Let *G* be a graph and *v*, *w* vertices in *G*.
- A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G: $v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$, where $v_0=v$, $v_n=w$ and v_{i-1} and v_i are endpoints of e_i.
- The trivial walk from *v* to *v* consists of a single vertex *v*.
- Note that if a graph does not contain parallel edges, then any walk is uniquely determined by its sequence of vertices.

Definitions

- A trail from *v* to *w* is a walk from *v* to *w* without a repeated edge.
- A path from *v* to *w* is a trail without a repeated vertex.
- A closed walk is a walk that starts and ends at the same vertex.
- A circuit is a closed walk that contains at least an edge but no repeated edge.
- A simple circuit is a circuit that does not contain any other repeated vertex except for the first and last.

Connectedness

- Two vertices *v* and *w* of a graph *G* are connected iff there is a walk from *v* to *w*.
- The graph G is connected iff all vertices in *G* are pairwise connected.
- A graph H is a connected component of a graph G iff
  - H is a subgraph of G
  - H is connected and no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H

Euler Circuit

- An Euler Circuit of a graph *G* is a circuit containing every vertex and every edge of G.
- If a graph has an Euler circuit, then every vertex of the graph has positive even degree.
- Contrapositive: If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

Euler Circuit

- Converse: If every vertex of a graph has even degree, then the graph has an Euler circuit.
- Consider graphs that are not connected.
- If every vertex of a connected graph has even degree, then the graph has an Euler circuit.

7

Constructive Proof

1. Pick a start vertex *v*.
2. Pick any sequence of distinct adjacent vertices and edges, starting and ending at *v*. Call the resulting circuit C.
3. If C contains every edge and vertex of G, we are done.
4. Otherwise

8

Constructive Proof

4. Remove all edges of *C* and any vertices that become isolated from *G*. Call the resulting graph *G'*.
5. Pick any *w* common to both *C* and *G'*.
6. Repeat step 2 on *w* and *G'*, resulting in circuit *C'* that starts and ends at *w*.
7. Combining *C* and *C'* results in a larger circuit that starts and ends in *v*.
8. Repeat until the graph is exhausted.

9

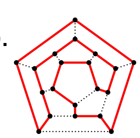
Theorem

- A graph G has an Euler circuit iff G is connected and every vertex of G has even degree.
- A graph G has an Euler trail from v to w iff G is connected, v and w have odd degree and all other vertices of G have positive even degree.

10

Hamiltonian Circuit

- What if we require that a circuit visit every vertex only once?
- The Hamiltonian puzzle (1859).



- A Hamiltonian circuit of a graph *G* is a simple circuit that includes every vertex of *G*.

11

Euler and Hamiltonian

- Euler does not allow repeating edges.
- Hamiltonian does not allow repeating edges or vertices (except for first and last).
- An Euler circuit includes every vertex of a graph, but may visit them more than once.
- A Hamiltonian circuit doesn't need to include every edge of a graph.
- Thus an Euler circuit may not be a Hamiltonian, and a Hamiltonian may not be an Euler.

12

Finding a Hamiltonian Circuit

- In general, a Hamiltonian circuit (if there is one in *G*), is a subgraph of *G*.
- There is no known efficient way to determine whether a graph has a Hamiltonian circuit, or how to find one.
- The Traveling Salesman Problem (TSP): a sales man wishes to visit each city once and only once, and minimize traveling distances.

Checking for No

- If a graph *G* has a Hamiltonian circuit, then *G* has a subgraph *H* with the following properties:
  - *H* contains every vertex of *G*
  - *H* is connected
  - *H* has the same number of edges as vertices
  - Every vertex of *H* has degree 2