Conditional Probability

CS231
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Boy or Girl?

- A couple has two children, one of them is a girl. What is the probability that the other one is also a girl? Assuming 50/50 chances of conceiving boys and girls.

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Conditional Probability

- Let A and B be events in a sample space S. If $P(A) \neq 0$, then the *conditional probability* of B given A ($P(B|A)$) is:

$$P(B|A) = P(A \cap B) / P(A)$$

$$P(A) = P(A \cap B) / P(B|A)$$
$$P(A \cap B) = P(B|A) \times P(A)$$

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Example

- Two cards are drawn from a well-shuffled deck. What is the probability that:
  - both are kings?
  - second draw is a king?
- A = 1st draw is king, B = 2nd draw is king
- $P(A) = 4/52$, $P(A^c) = 48/52$
- $P(B|A) = 3/51$, $P(B|A^c) = 4/51$
- $P(A \cap B) = 4/52 \times 3/51 = 12/2652$
- $P(A \cap B) + P(A^c \cap B) = 4/52 \times 3/51 + 48/52 \times 4/51$

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Example

- If the experiment of drawing a pair is repeated over time, what would be the expected value of the number of of kings?
- 2 kings: $P(A \cap B) = 4/52 \times 3/51 = 12/2652$
- 1 king: $= P(A^c \cap B) + P(A \cap B^c) = 48/52 \times 4/51 + 4/52 \times 48/51 = 384/2652$
- Expected value of # of kings: $2 \times 12/2652 + 1 \times 384/2652 \approx 0.154$

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Example

- 5% of manufactured components are defective in general.
- The method for screening out defective items is not totally reliable. The test rejects good parts as defective in 1% of the cases and accepts defective parts as good ones in 10% of the cases.
- Given that the test indicates that an item is good, what is the probability that this item is, in fact, defective?

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Definitions

- T = A component tested good
- D = A component is defective
- T^c = A component tested defective
- G = A component is good ($G = D^c$)

- Want to solve: $P(D|T)$

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$P(D|T)$

- $P(D) = 0.05$, $P(G) = 0.95$
- $P(T^c|G) = 0.01$ (false positive) $P(T|G) = 0.99$
- $P(T \cap G) = P(T|G) \times P(G) = 0.99 \times 0.95 = 0.9405$
- $P(T|D) = 0.1$ (false negative)
- $P(T \cap D) = P(T|D) \times P(D) = 0.1 \times 0.05 = 0.005$

- $T = (T \cap G) \cup (T \cap D)$
- $P(T) = 0.9405 + 0.005 = 0.9455$
- $P(D|T) = P(T \cap D)/P(T) = 0.005/0.9455 = 0.0052882$

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Medical Screening

- 1% of population suffer from a certain disease.
- The method for screening is not totally reliable. The test reports false positive in 5% of the cases and false negative in 10% of the cases.
- Given that a person has a negative test result, what is the probability that this person is, in fact, sick?
- Given that a person has a positive test result, what is the probability that this person is, in fact, sick?

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Definitions

- T = A person cleared the test (negative)
- S = A person is sick
- T^c = A person did not clear the test (positive)
- H = A person is healthy ($H = S^c$)

- Want to solve: $P(S|T)$ and $P(S|T^c)$

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$P(S|T)$

- $P(S) = 0.01$, $P(H) = 0.99$
- $P(T^c|H) = 0.05$ (false positive) $P(T|H) = 0.95$
- $P(T \cap H) = P(T|H) \times P(H) = 0.95 \times 0.99 = 0.9405$
- $P(T|S) = 0.1$ (false negative)
- $P(T \cap S) = P(T|S) \times P(S) = 0.1 \times 0.01 = 0.001$

- $T = (T \cap H) \cup (T \cap S)$
- $P(T) = 0.9405 + 0.001 = 0.9415$
- $P(S|T) = P(T \cap S)/P(T) = 0.001/0.9415 \approx 0.001$

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Bayes' Theorem

- If $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events in a sample space, the total probability of any event F is:

$$P(F) = \sum_{i=1}^{n} P(F \mid E_i) P(E_i)$$

- For any event E and F with $P(F) \neq 0$:

$$P(E \mid F) = \frac{P(F \mid E) P(E)}{P(F \mid E) P(E) + P(F \mid \overline{E}) P(\overline{E})}$$

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P(S|T) and P(S|T^c) with Bayes'

- P(S) = 0.01, P(H) = P(S^c) = 0.99
- P(T^c|H) = P(T^c|S^c) = 0.05, P(T|H) = P(T|S^c) = 0.95
- P(T|S) = 0.1, P(T^c|S) = 0.9

$$P(S \mid T) = \frac{P(T \mid S)P(S)}{P(T \mid S)P(S) + P(T \mid \overline{S})P(\overline{S})}$$

$$= \frac{0.1 \times 0.01}{0.1 \times 0.01 + 0.95 \times 0.99} = \frac{0.001}{0.001 + 0.9405} \approx 0.001$$

$$P(S \mid \overline{T}) = \frac{P(\overline{T} \mid S)P(S)}{P(\overline{T} \mid S)P(S) + P(\overline{T} \mid \overline{S})P(\overline{S})}$$

$$= \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.05 \times 0.99} = \frac{0.009}{0.009 + 0.0495} \approx 0.1538$$

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Bayes' Theorem

- If E_1, E_2, ..., E_n are mutually exclusive and exhaustive events in a sample space, given any F with P(F) ≠ 0,

$$P(E_k \mid F) = \frac{P(F \mid E_k)P(E_k)}{\sum_{i=1}^{n} P(F \mid E_i)P(E_i)} = \frac{P(F \mid E_k)P(E_k)}{P(F)}$$

- P(θ) is also known as prior
- P(θ|X) is the posterior probability after observing X and obtaining P(X|θ)

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Monty Hall Revisited

- A = prize is behind door A, B = prize is behind door B, C = prize is behind door C
- M_A = Monty opens door A, etc
- You choose door A and Monty opens a door revealing no prize
- P(M_B|A) = 1/2, P(M_B|B) = 0, P(M_B|C) = 1
- P(M_B) = 1/2, P(A) = P(B) = P(C) = 1/3
- P(A|M_B) = P(M_B|A)P(A)/P(M_B) = 1/2 × 1/3 / 1/2 = 1/3
- P(C|M_B) = P(M_B|C)P(C)/P(M_B) = 1 × 1/3 / 1/2 = 2/3
- Exact same analysis holds for M_C

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Bayes' Ratio

- When there are three events, A, B and C and the comparative posterior probabilities are of interest, consider the ratio:

$$\frac{P(A \mid C)}{P(B \mid C)} = \frac{P(C \mid A)}{P(C \mid B)} \times \frac{P(A)}{P(B)}$$

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Example

- Two bags, one contains 70 red and 30 blue balls, and the other 30 red and 70 blue balls.
- Choose one bag randomly and draw with replacement.
- 8 red and 4 blue balls are drawn in 12 tries.
- What is the probability that it was the predominantly red bag that was chosen?

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Solution

- A = selecting the 1st bag, B = selecting the 2nd bag, C = getting the draws we did
- P(C|A) = $(7/10)^8$x$(3/10)^4$xC(12,8)
- P(C|B) = $(7/10)^4$x$(3/10)^8$xC(12,8)
- P(A) = P(B) = 0.5
- P(C|A) / P(C|B) = $(7/10)^4/(3/10)^4$ = $(7/3)^4$
- P(A|C) / P(B|C) = $(7/3)^4$
- P(A|C) + P(B|C) = 1

} P(A|C) = $(7/3)^4$ / $((7/3)^4+1)$

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Dramatic Taxicab

- A cab was involved in a hit-and-run at night.
- Two cab companies operate in the city, with green and blue cabs, respectively.
- 85% of the cabs are green.
- A witness identified the cab as blue.
- The witness correctly identified the two colors 80% of the time under night-time testing.
- What is the probability that the witness was right?

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Independent Events

- Two events are independent when the occurrence of one does not affect the probability of the other.
  - tossing coins
  - rolling dice
- Events *A* and *B* are independent iff:

$$P(A \cap B) = P(A) \times P(B)$$

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$P(A \cap B^c)$

- If *A* and *B* are independent events, so are *A* and B^c.
- From set theory:
  - $(A \cap B) \cup (A \cap B^c) = A$
  - $(A \cap B) \cap (A \cap B^c) = \varnothing$
- $P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c) = P(A)$
- $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$

$= P(A)(1 - P(B)) = P(A)P(B^c)$

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Loaded Coin

- A coin is loaded so that the probability of heads is 0.6. After 10 tosses, what is the probability of obtaining 8 heads?
- Consider HHHHHHHHTT
- $P(HHHHHHHHTT) = 0.6^8 \times 0.4^2$
- How many ways can you get 8 heads with 10 tosses? – C(10, 8)
- $P(8 \text{ heads}) = C(10, 8) \times 0.6^8 \times 0.4^2 \approx 0.12$

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