Gambling and Probability

CS231

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The game of poker

- You are given 5 cards (this is 5-card stud poker)
- The goal is to obtain the best hand you can
- The possible poker hands are (in increasing order):
  - No pair
  - One pair (two cards of the same face)
  - Two pair (two sets of two cards of the same face)
  - Three of a kind (three cards of the same face)
  - Straight (all five cards sequentially – ace is either high or low)
  - Flush (all five cards of the same suit)
  - Full house (a three of a kind of one face and a pair of another face)
  - Four of a kind (four cards of the same face)
  - Straight flush (both a straight and a flush)
  - Royal flush (a straight flush that is 10, J, K, Q, A)

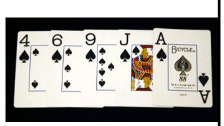
Poker probability: royal flush

- What is the chance of getting a royal flush?
  - 10, J, Q, K, and A of the same suit
- There are only 4 possible royal flushes
- Cardinality for 5-cards: C(52,5) = 2,598,960
- Probability = 4/2,598,960 = 0.0000015
  - Or about 1 in 650,000

Poker probability: four of a kind

- What is the chance of getting 4 of a kind when dealt 5 cards?
  - 5 cards: C(52,5) = 2,598,960
- Possible hands that have four of a kind:
  - There are 13 possible four of a kind hands
  - The fifth card can be any of the remaining 48 cards
  - Thus, total is 13*48 = 624
- Probability = 624/2,598,960 = 0.00024
  - Or 1 in 4165

Poker probability: flush



- What is the chance of a flush?
  - 5 cards of the same suit
- We must do ALL of the following:
  - Pick the suit for the flush: C(4,1)
  - Pick the 5 cards in that suit: C(13,5)
- Product rule: C(13, 5)*C(4, 1) = 5148
- Probability = 5148/2,598,960 = 0.00198
  - Or about 1 in 505
- Note that if you don't count straight flushes (and thus royal flushes) as a "flush", then the number is really 5108

Poker probability: full house

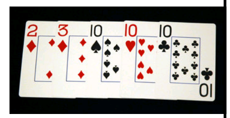
- What is the chance of getting a full house?
  - Three cards of one face and two of another face
- We must do ALL of the following:
  - Pick the face for the three of a kind: C(13,1)
  - Pick the 3 of the 4 cards to be used: C(4,3)
  - Pick the face for the pair: C(12,1)
  - Pick the 2 of the 4 cards of the pair: C(4,2)
- C(13, 1)*C(4, 3)*C(12, 1)*C(4, 2) = 3744
- Probability = 3744/2,598,960 = 0.00144
  - Or about 1 in 694

Inclusion-exclusion principle

- The possible poker hands are (in increasing order):

– Nothing	
– One pair	cannot include two pair, three of a kind, four of a kind, or full house
– Two pair	cannot include three of a kind, four of a kind, or full house
– Three of a kind	cannot include four of a kind or full house
– Straight	cannot include straight flush or royal flush
– Flush	cannot include straight flush or royal flush
– Full house	
– Four of a kind	
– Straight flush	cannot include royal flush
– Royal flush	

Poker probability: three of a kind



- What is the chance of getting a three of a kind?
  - That's three cards of one face
  - Can't include a full house or four of a kind
- We must do ALL of the following:
  - Pick the face for the three of a kind: C(13,1)
  - Pick the 3 of the 4 cards to be used: C(4,3)
  - Pick the two other cards' face values: C(12,2)
  - Pick the suits for the two other cards: C(4,1)*C(4,1)

- C(13,1)*C(4,3)*C(12,2)*C(4,1)*C(4,1) = 54912
- Probability = 54,912/2,598,960 = 0.0211
  - Or about 1 in 47

Poker hand odds

- The possible poker hands are (in increasing order):

– Nothing	1,302,540	0.5012
– One pair	1,098,240	0.4226
– Two pair	123,552	0.0475
– Three of a kind	54,912	0.0211
– Straight	10,200	0.00392
– Flush	5,108	0.00197
– Full house	3,744	0.00144
– Four of a kind	624	0.000240
– Straight flush	36	0.0000139
– Royal flush	4	0.00000154

Probability axioms

- Let E be an event in a sample space S. The probability of the complement of E is:

$$p(\overline{E}) = 1 - p(E)$$

- Recall the probability for getting a royal flush is 0.0000015
  - The probability of *not* getting a royal flush is 1-0.0000015 or 0.9999985
- Recall the probability for getting a four of a kind is 0.00024
  - The probability of *not* getting a four of a kind is 1-0.00024 or 0.99976

Probability of the union of two events

- Let E_1 and E_2 be events in sample space S

- Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Probability of the union of two events

- If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?
- Let n be the number chosen
  - $p(2|n)$ = 50/100 (all the even numbers)
  - $p(5|n)$ = 20/100
  - $p(2|n)$ and $p(5|n)$ = $p(10|n)$ = 10/100
  - $p(2|n)$ or $p(5|n)$ = $p(2|n) + p(5|n) - p(10|n)$
    = 50/100 + 20/100 – 10/100
    = 3/5

When is gambling worth it?

- This is a *statistical* analysis, not a moral/ethical discussion
- What if you gamble $1, and have a ½ probability to win $10?
- What if you gamble $1 and have a 1/100 probability to win $10?
- One way to determine if gambling is worth it:
  - probability of winning * payout ≥ amount spent per play

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Expected Value

- The expected values of a process with outcomes of values $a_1, a_2, \ldots, a_n$ which occur with probabilities $p_1, p_2, \ldots, p_n$ is:

$$\sum_{i=1}^{n} a_i p_i$$

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Expected values of gambling

- Gamble $1, and have a ½ probability to win $10
  - (10-1)*0.5+(-1)*0.5 = 4
- Gamble $1 and have a 1/100 probability to win $10?
  - (10-1)*0.01+(-1)*0.99 = -0.9
- Another way to determine if gambling is worth it: Expected value > 0

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When is lotto worth it?

- In many older lotto games (Pick-6) you have to choose 6 numbers from 1 to 48
  - Total possible choices are C(48,6) = 12,271,512
  - Total possible winning numbers is C(6,6) = 1
  - Probability of winning is 0.0000000814
    - Or 1 in 12.3 million
- If you invest $1 per ticket, it is only statistically worth it if the payout is > $12.3 million

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Powerball lottery

- Modern powerball lottery: you pick 5 numbers from 1-55
  - Total possibilities: C(55,5) = 3,478,761
- You then pick one number from 1-42 (the powerball)
  - Total possibilities: C(42,1) = 42
- You need to do both -- apply the product rule,
  - Total possibilities are 3,478,761* 42 = 146,107,962
- While there are many "sub" prizes, the probability for the jackpot is about 1 in 146 million
- If you count in the other prizes, then you will "break even" if the jackpot is $121M

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Blackjack

- You are initially dealt two cards
  - 10, J, Q and K all count as 10
  - Ace is EITHER 1 or 11 (player's choice)
- You can opt to receive more cards (a "hit")
- You want to get as close to 21 as you can
  - If you go over, you lose (a "bust")
- You play against the house
  - If the house has a higher score than you, then you lose

Blackjack table

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Blackjack probabilities

- Getting 21 on the first two cards is called a blackjack
  - Or a "natural 21"
- Assume there is only 1 deck of cards
- Possible blackjack blackjack hands:
  - First card is an A, second card is a 10, J, Q, or K
    - 4/52 for Ace, 16/51 for the ten card
    - = (4*16)/(52*51) = 0.0241 (or about 1 in 41)
  - First card is a 10, J, Q, or K; second card is an A
    - 16/52 for the ten card, 4/51 for Ace
    - = (16*4)/(52*51) = 0.0241 (or about 1 in 41)
- Total chance of getting a blackjack is the sum of the two:
  - p = 0.0483, or about 1 in 21
  - More specifically, it's 1 in 20.72

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Blackjack probabilities

- Another way to get 20.72
- There are C(52,2) = 1,326 possible initial blackjack hands
- Possible blackjack blackjack hands:
  - Pick your Ace: C(4,1)
  - Pick your 10 card: C(16,1)
  - Total possibilities is the product of the two (64)
- Probability is 64/1,326 = 1 in 20.72 (0.048)

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Blackjack probabilities

- Assume there is an infinite deck of cards
- Possible blackjack blackjack hands:
  - First card is an A, second card is a 10, J, Q, or K
    - 4/52 for Ace, 16/52 for second part
    - = (4*16)/(52*52) = 0.0236 (or about 1 in 42)
  - First card is a 10, J, Q, or K; second card is an A
    - 16/52 for first part, 4/52 for Ace
    - = (16*4)/(52*52) = 0.0236 (or about 1 in 42)
- Total chance of getting a blackjack is the sum:
  - p = 0.0473, or about 1 in 21
  - More specifically, it's 1 in 21.13 (vs. 20.72)
- In reality, most casinos use "shoes" of 6-8 decks for this reason
  - It slightly lowers the player's chances of getting a blackjack
  - And prevents people from counting the cards…

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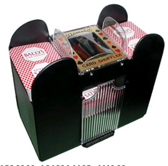
Counting cards and Continuous Shuffling Machines (CSMs)

- Counting cards means keeping track of which cards have been dealt, and how that modifies the chances
- After cards are discarded, they are added to the continuous shuffling machine
- Many blackjack players refuse to play at a casino with one
  - So they aren't used as much as casinos would like



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So always use a single deck, right?

- Most people think that a single-deck blackjack table is better, as the player's odds increase
  - And you can try to count the cards
- Normal rules have a 3:2 payout for a blackjack
  - If you bet $100, you get your $100 back plus 3/2 * $100, or $150 additional
- Most single-deck tables have a 6:5 payout
  - You get your $100 back plus 6/5 * $100 or $120 additional
  - The expected value of the game is lowered
  - This OUTWEIGHS the benefit of the single deck!
    - And the benefit of counting the cards
  - Remember, the house always wins

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Buying (blackjack) insurance

- If the dealer's visible card is an Ace, the player can buy insurance against the dealer having a blackjack
  - There are then two bets going: the original bet and the insurance bet
  - If the dealer has blackjack, you lose your original bet, but your insurance bet pays 2-to-1
    - So you get twice what you paid in insurance back
    - Note that if the player also has a blackjack, it's a "push"
  - If the dealer does not have blackjack, you lose your insurance bet, but your original bet proceeds normal
- Is this insurance worth it?

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Buying (blackjack) insurance

- If the dealer shows an Ace, there is a 16/52 = 0.308 probability that they have a blackjack
  - Assuming an infinite deck of cards
  - Any one of the "10" cards will cause a blackjack
- If you bought insurance 1,000 times, it would be used 308 (on average) of those times
  - Let's say you paid $1 each time for the insurance
- The payout on each is 2-to-1, thus you get $2 back when you use your insurance
  - Thus, you get 2*308 = $616 back for your $1,000 spent
- Or, using the formula p(winning) * payout ≥ investment
  - 0.308 * $2 ≥ $1?
  - 0.616 ≥ $1?
  - Thus, it's not worth it

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Why counting cards doesn't work well…

- If you make two or three mistakes an hour, you lose any advantage
  - And, in fact, cause a disadvantage!
- You lose lots of money learning to count cards
- Then, once you can do so, you are banned from the casinos

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So why is Blackjack so popular?

- Although the casino has the upper hand, the odds are much closer to 50-50 than with other games
  - Players following strategy will lose less than 1% on average luck
- Notable exceptions are games that you are not playing against the house – i.e., poker

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Blackjack Strategy Chart

Your Hand	Dealer's Up Card									
Total	2	3	4	5	6	7	8	9	10	A
5-8	H	H	H	H	H	H	H	H	H	H
9	H	D	D	D	D	H	H	H	H	H
10	D	D	D	D	D	D	D	D	H	H
11	D	D	D	D	D	D	D	D	D	H
12	H	H	S	S	S	H	H	H	H	H
13,14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	H	H
16	S	S	S	S	S	H	H	H	H	H
17-21	S	S	S	S	S	S	S	S	S	S
A, 2-3	H	H	H	D	D	H	H	H	H	H
A, 4-5	H	H	D	D	D	H	H	H	H	H
A,6	H	D	D	D	D	H	H	H	H	H
A,7	S	D	D	D	D	S	S	H	H	H
A,8-9	S	S	S	S	S	S	S	S	S	S
2,2 / 3,3	P	P	P	P	P	P	H	H	H	H
4,4	H	H	H	P	P	H	H	H	H	H
5,5	D	D	D	D	D	D	D	D	H	H
6,6	P	P	P	P	P	H	H	H	H	H
7,7	P	P	P	P	P	P	H	H	H	H
8,	P	P	P	P	P	P	P	P	P	P
9,9	P	P	P	P	P	S	P	P	S	S
10,10	S	S	S	S	S	S	S	S	S	S
A,A	P	P	P	P	P	P	P	P	P	P

Legend				
	H	Hit	D	Double
	S	Stand	P	Split

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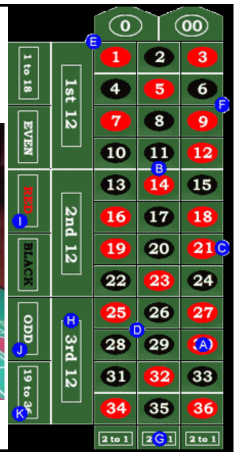
Roulette

- A wheel with 38 spots is spun
  - Spots are numbered 1-36, 0, and 00
  - European casinos don't have the 00
- A ball drops into one of the 38 spots
- A bet is placed as to which spot or spots the ball will fall into
  - Money is then paid out if the ball lands in the spot(s) you bet upon

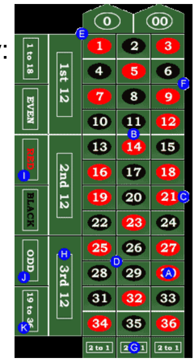
The Roulette table



The Roulette table

- Bets can be placed on:

Bet	Probability:
– A single number	1/38
– Two numbers	2/38
– Four numbers	4/38
– All even numbers	18/38
– All odd numbers	18/38
– The first 18 nums	18/38
– Red numbers	18/38



The Roulette table

- Bets can be placed on:

Bet	Probability:	Payout:
– A single number	1/38	36*x*
– Two numbers	2/38	18*x*
– Four numbers	4/38	9*x*
– All even numbers	18/38	2*x*
– All odd numbers	18/38	2*x*
– The first 18 nums	18/38	2*x*
– Red numbers	18/38	2*x*

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Roulette

- It has been proven that proven that no advantageous strategies exist
- Including:
  - Learning the wheel's biases
    - Casinos regularly balance their Roulette wheels
  - Using lasers (yes, lasers) to check the wheel's spin
    - What casino will let you set up a laser inside to beat the house?

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Roulette

- Martingale betting strategy
  - Where you double your (outside) bet each time (thus making up for all previous losses)
  - It still won't work!
  - You can't double your money forever
    - It could easily take 50 times to achieve a final win
    - If you start with \$1, then you must put in $\$1*2^{50}$ = \$1,125,899,906,842,624 to win this way!
    - That's 1 quadrillion

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