The Binomial Theorem

CS231

Dianna Xu

Combinatorial Proof

- A *combinatorial proof* is a proof that uses counting arguments to prove a theorem
  - Rather than some other method such as algebraic techniques
- Essentially, show that both sides of the proof manage to count the same objects
- In other words, a bijection between the two sets

Pascal's Formula

- One of the most famous and useful in Combinatorics
- C(n+1, r) = C(n, r-1) + C(n, r)
- Recall another important combinatorial result:
- C(n, r) = C(n, n-r)

Combinatorial Proof

- C(n+1, r):
  - # of ways to choose r elements from n+1
- Remove an arbitrary element from n+1, call it a.
- Now form all possible subsets of size r.
  - Theses are all the subsets of size r you can have without a.
  - C(n, r)
- Now we need to account for subsets of size r with a

Combinatorial Proof

- Now we need to account for subsets of size r with a
  - From the same n elements, form all possible subsets of size r-1, then add a.
  - C(n, r-1)

Algebraic Proof

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$$

$$\frac{(n+1)n!}{k(k-1)!(n+1-k)(n-k)!} = \frac{n!}{(k-1)!(n-k+1)(n-k)!} + \frac{n!}{k(k-1)!(n-k)!}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{1}{(n-k+1)} + \frac{1}{k}$$

$$\frac{(n+1)}{k(n+1-k)} = \frac{k}{k(n-k+1)} + \frac{(n-k+1)}{k(n-k+1)}$$

$$n+1 = k+n-k+1$$

$$n+1 = n+1$$

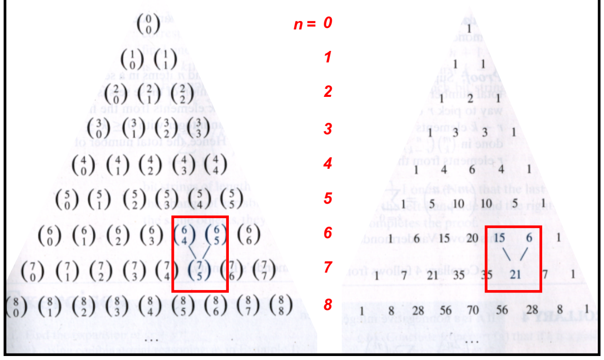
Substitutions:

$$(n-k+1)! = (n-k+1)(n-k)!$$

$$(n+1)! = (n+1)n!$$

$$k! = k(k-1)!$$

Pascal's triangle



Binomial Coefficients

- A quick expansion of $(x+y)^n$
- Why it's really important:
- It provides a good context to present proofs
  - Especially combinatorial proofs

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Polynomial Expansion

- Consider $(x+y)^3$: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- Rephrase it as:

$$(x+y)(x+y)(x+y) = x^3 + \left[x^2y + x^2y + x^2y\right] + \left[xy^2 + xy^2 + xy^2\right] + y^3$$

- When choosing x twice and y once, there are C(3,2) = C(3,1) = 3 ways to choose where the x comes from
- When choosing x once and y twice, there are C(3,2) = C(3,1) = 3 ways to choose where the y comes from

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Polynomial expansion

- Consider $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
- To obtain the x^5 term
  - Each time you multiple by $(x+y)$, you select the x
  - Thus, of the 5 choices, you choose x 5 times or y 0 times
    - C(5,5) = 1 = C(5, 0)
- To obtain the x^4y term
  - Four of the times you multiply by $(x+y)$, you select the x
    - The other time you select the y
  - Thus, of the 5 choices, you choose x 4 times or y 1 time
    - C(5,4) = 5 = C(5, 1)
- To obtain the x^3y^2 term
  - C(5,3) = C(5,2) = 10

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Polynomial expansion

- For $(x+y)^5$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$

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Polynomial Expansion: The Binomial Theorem

- For $(x+y)^n$

$$(x+y)^n = \binom{n}{n}x^ny^0 + \binom{n}{n-1}x^{n-1}y^1 + \cdots + \binom{n}{1}x^1y^{n-1} + \binom{n}{0}x^0y^n$$

$$= \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

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Sample question

- Find the coefficient of x^5y^8 in $(x+y)^{13}$
- Answer: $\binom{13}{5}=\binom{13}{8}=1287$

Examples

- What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

$$\binom{25}{12}=\binom{25}{13}=\frac{25!}{13!12!}=5,200,300$$

- What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?

$$(2x+(-3y))^{25}=\sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^j$$

  - The coefficient occurs when j=13:

$$\binom{25}{13}2^{12}(-3)^{13}=\frac{25!}{13!12!}2^{12}(-3)^{13}=-33,959,763,545,702,400$$

Pascal's Triangle

n		sum = 2^n
0	1	1
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
	…	

Corollary 1 and Algebraic Proof

$$\sum_{k=0}^{n}\binom{n}{k}=2^n, n\geq 0$$

- Algebraic proof

$$2^n=(1+1)^n \rightarrow (x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

$$=\sum_{k=0}^{n}\binom{n}{k}1^k1^{n-k}$$

$$=\sum_{k=0}^{n}\binom{n}{k}$$

Combinatorial Proof $\sum_{k=0}^{n}\binom{n}{k}=2^n, n\geq 0$

- A set with n elements has 2^n subsets
  - By definition of and cardinality of power set
- Each subset has either 0 or 1 or 2 or … or n elements
  - There are $\binom{n}{0}$ subsets with 0 elements,
  - $\binom{n}{1}$ subsets with 1 element, …
  - and $\binom{n}{n}$ subsets with n elements
  - Thus, the total number of subsets is $\sum_{k=0}^{n}\binom{n}{k}$

Pascal's Triangle

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
	…

Corollary 2

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0, n\geq 1$$

- Algebraic proof $0=0^n$

$$=((-1)+1)^n \quad (x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$
$$=\sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}(-1)^k$$

- This implies that

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

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Corollary 3

- Let n be a non-negative integer. Then

$$\sum_{k=0}^{n}2^k\binom{n}{k}=3^n$$

- Algebraic proof

$$3^n=(1+2)^n$$
$$=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}2^k$$
$$=\sum_{k=0}^{n}\binom{n}{k}2^k$$

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Vandermonde's identity

- Let m, n, and r be non-negative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$$

- Assume a congressional committee must consist of r people, and there are n Democrats and m Republicans
  - How many ways are there to pick the committee?

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Combinatorial proof of Vandermonde's identity $\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$

- Consider two sets, one with m items and one with n items
  - Then there are $\binom{m+n}{r}$ ways to choose r items from the union of those two sets
- Next, we find that value via a different means
  - Pick k elements from the set with n elements
  - Pick the remaining r-k elements from the set with m elements
  - Via the product rule, there are $\binom{m}{r-k}\binom{n}{k}$ ways to do that for **EACH** value of k
  - Lastly, consider this for all values of k: $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$
- Thus, $\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$

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More Combinatorial Proofs

- $n^3 - n = 6C(n,2) + 6C(n,3)$
- $n^3 - n = (n+1)n(n-1)$
- $= n(n-1)(n-2) + 3n(n-1)$
- $n^3 - n = P(n+1, 3)$

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