Sets

CS 231
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What is a set?

- A set is a group of "objects"
  - People: {Alice, Bob, Clara}
  - Colors of a rainbow: {red, orange, yellow, green, blue, purple}
  - States in the US: {Alabama, Alaska, Virginia, …}
  - All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}
  - Unrelated elements: {3, a, red, Virginia}

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Notes

- Order does not matter
  - We often write them in order because it is easier for humans to understand it that way
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}
- Sets are notated with curly brackets
- No duplicate elements
- Note that a list is like a set, but order does matter and duplicate elements are allowed

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Specifying a Set

- List all the elements: A = {1, 2, 3, 4, 5}
  - Not always feasible for large or infinite sets
- Set-builder notation
  - D = {x | x is prime and $x > 2$}
  - all elements x such that x is prime and x is greater than 2
  - E = {$x \in Z$ | $x > 2$}
  - all integers x such that x is greater than 2

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Specifying a Set

- A set is said to "contain" the various "members" or "elements" that make up the set
  - If an element a is a member of (or an element of) a set S, we use then notation $a \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
    - $7 \notin \{1, 2, 3, 4\}$
   - Virginia $\notin \{1, 2, 3, 4\}$

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The Universal Set

- U is the universal set – the set of all of elements (or the "universe") from which given any set is drawn
  - For the set {-2, 0.4, 2}, U would be the real numbers
  - For the set {0, 1, 2}, U could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context

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Venn diagrams

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram

```
b  c  d  f              U
g  h  j        S
k  l  m
n  p  q     a   e   i
r  s  t
v  w  x        o   u
y  z
```
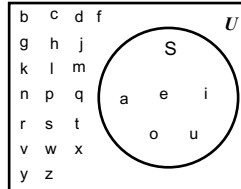
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Sets of sets

- Sets can contain other sets
  - S = { {1}, {2}, {3} }
  - T = { {1}, {{2}}, {{{3}}} }
  - V = { {{1}, {{2}}, {{{3}}}, {1}, {{2}}, {{{3}}} }
    - V has only 3 elements!
- Note that $1 \neq \{1\} \neq \{\{1\}\} \neq \{\{\{1\}\}\}$
  - They are all different

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The Empty Set

- If a set has zero elements, it is called the empty (or null) set
  - Written using the symbol $\varnothing$
  - Thus, $\varnothing = \{\ \}$
- As the empty set is a set, it can be a element of other sets
  - $\{ \varnothing, 1, 2, 3, x \}$ is a valid set

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The Empty Set

- Note that $\varnothing \neq \{ \varnothing \}$
  - The first is a set of zero elements
  - The second is a set of 1 element (that one element being the empty set)

- Replace $\varnothing$ by $\{\ \}$, and you get: $\{\ \} \neq \{\ \{\ \}\ \}$
    - It's easier to see that they are not equal that way

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Set Equality

- Two sets are equal iff they have the same elements
  - $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
  - $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
  - $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

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Subsets

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}
  - This is specified by $S \subseteq T$
- If S is not a subset of T, it is written as such:
  $S \nsubseteq T$
  - For example, $\{1, 2, 8\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7\}$

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Subsets

- Any set is a subset of itself!
  - Since all the elements of S are elements of S, $S \subseteq S$
- The empty set is a subset of *all* sets (including itself!)
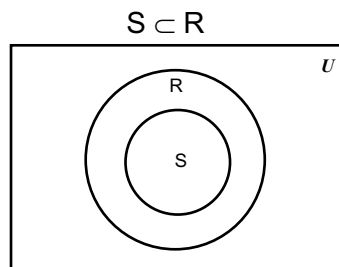- *All* sets are subsets of the universal set

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Proper Subsets

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - denoted as $S \subset T$
- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

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Proper Subsets: Venn Diagram

$S \subset R$

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Set Cardinality

- The cardinality of a set is the number of elements in a set
  - Written as |A|
- Examples
  - Let R = {1, 2, 3, 4, 5}. Then |R| = 5
  - $|\varnothing| = 0$
  - Let S = {∅, {a}, {b}, {a, b}}. Then |S| = 4
- A set with one element is sometimes called a singleton set

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Power Sets

- Given the set S = {0, 1}. What are all the possible subsets of S?
  - They are: ∅, {0}, {1}, and {0, 1}
- The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
    - Note that |S| = 2 and |P(S)| = 4

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Power Sets
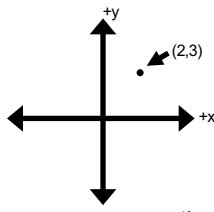
- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }
    - Note that |T| = 3 and |P(T)| = 8
- P(∅) = { ∅ }
    - Note that $|\varnothing| = 0$ and $|P(\varnothing)| = 1$
- If a set has *n* elements, then the power set will have 2^n elements

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Tuples

- In 2-dimensional space, it is an **ordered** pair of numbers (for example, to specify a location)

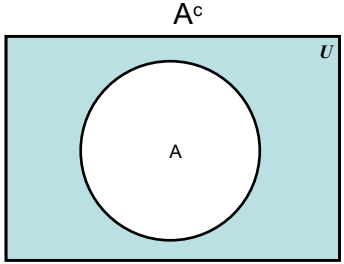- In n-dimensional space, it is a n-tuple of ordered numbers

+y

(2,3)

+x

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Cartesian Product

- A Cartesian product is a set of all ordered 2-tuples where each "part" is from a given set: A x B = { (a,b) | $a \in$ A and $b \in$ B }
- Given A = {a, b} and B = {0, 1}, what is their Cartesian product?
  - C = A x B = {(a,0), (a,1), (b,0), (b,1)}
- 2-D Cartesian coordinates are the set of all ordered pairs $\mathcal{Z}$x$\mathcal{Z}$ (or $\mathcal{R}$x$\mathcal{R}$)
- A 3-D coordinate is an element from the Cartesian product of $\mathcal{Z}$x$\mathcal{Z}$x$\mathcal{Z}$ (or $\mathcal{R}$x$\mathcal{R}$x$\mathcal{R}$)

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Complement

A^c

U

A

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Complement

A^{c}B^c

U

A   B
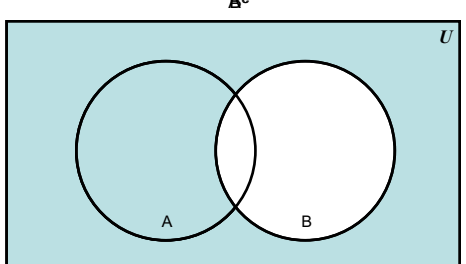
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Complement

- Complement of a set:
  $A^c = \{ x \mid x \notin A \}$
- Also written as $\overline{A}$
- Further examples (assuming $U = \mathcal{Z}$)
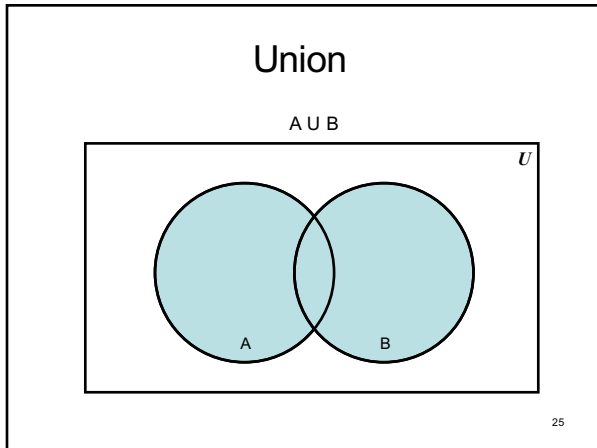 - {1, 2, 3}^{c} = { …, -2, -1, 0, 4, 5, 6, … }

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Complement

- Properties of complement sets

  - $(A^c)^c$ = A   Double Complement
  - A U A^c = U Complement law
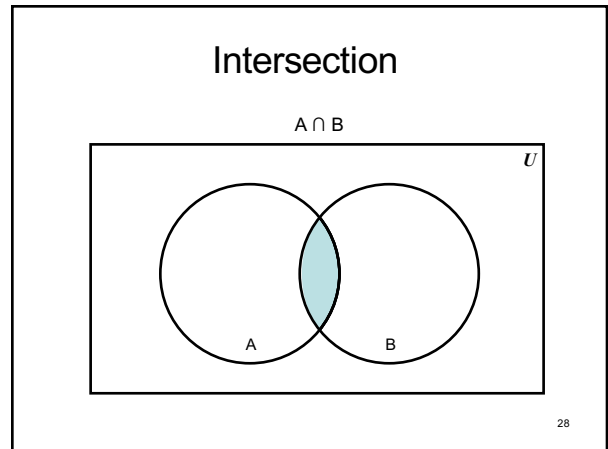 - A $\cap$ A^c = $\varnothing$ Complement law

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Union

A U B

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Union

- Union of two sets:
  A U B = { $x \mid x \in$ A $\vee$ $x \in$ B }
- Further examples
  – {1, 2, 3} U {3, 4, 5} = {1, 2, 3, 4, 5}
  – {New York, Washington} U {3, 4} = {New York, Washington, 3, 4}
  – {1, 2} U $\varnothing$ = {1, 2}

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Union Properties

- Properties of the union operation
  – A U $\varnothing$ = A   Identity law
  – A U U = U Universal Bound
  – A U A = A   Idempotent law
  – A U B = B U A   Commutative law
  – A U (B U C) = (A U B) U C   Associative law

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Intersection

A ∩ B

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Intersection

- Intersection of two sets:
  A ∩ B = { $x \mid x \in$ A $\wedge$ $x \in$ B }
- Further examples
  – {1, 2, 3} ∩ {3, 4, 5} = {3}
  – {New York, Washington} ∩ {3, 4} = $\varnothing$
    • No elements in common
  – {1, 2} ∩ $\varnothing$ = $\varnothing$
    • Any set intersection with the empty set yields the empty set

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Intersection Properties

- Properties of the intersection operation
  – A ∩ U = A   Identity law
  – A ∩ $\varnothing$ = $\varnothing$ Universal Bound
  – A ∩ A = A   Idempotent law
  – A ∩ B = B ∩ A   Commutative law
  – A ∩ (B ∩ C) = (A ∩ B) ∩ C   Associative law

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Disjoint Sets
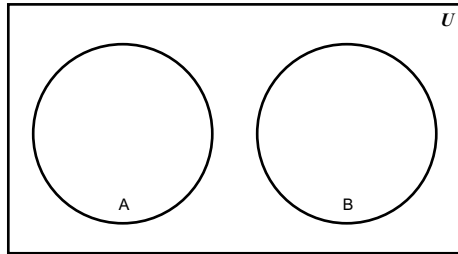
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Disjoint sets

- Disjoint sets: two sets are disjoint if their intersection is the empty set
- Further examples
  - {1, 2, 3} and {3, 4, 5} are not disjoint
  - {New York, Washington} and {3, 4} are disjoint
  - {1, 2} and $\varnothing$ are disjoint
    - Their intersection is the empty set
  - $\varnothing$ and $\varnothing$ are disjoint!
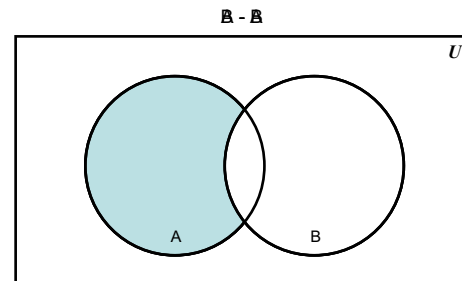   - Their intersection is the empty set

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Mutually Disjoint

- Sets A_1, A_2, …, A_n are mutually disjoint iff no two distinct sets A_i and A_j have any elements in common, in other words, $A_i \cap A_j = \varnothing$.
- A collection of nonempty sets {A_1, A_2, A_3 …} is as partition of a set $\mathcal{A}$ iff,
  - $\mathcal{A}$ is the union of all A_i
  - A_1, A_2, A_3, … are mutually disjoint

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Difference

A - B

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Difference

- Difference of two sets:
  $A - B = \{\, x \mid x \in A \land x \notin B \,\}$
  $A - B = A \cap B^c$ ← Important!
- Further examples
  - {1, 2, 3} - {3, 4, 5} = {1, 2}
  - {New York, Washington} - {3, 4} = {New York, Washington}
  - {1, 2} - $\varnothing$ = {1, 2}
  - The difference of any set S with $\varnothing$ is S

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Symmetric Difference

- Symmetric difference of two sets:
  $A \oplus B = \{\, x \mid (x \in A \lor x \in B) \land x \notin A \cap B\}$
  $A \oplus B = (A \cup B) - (A \cap B)$ ← Important!
- Further examples
  - {1, 2, 3} $\oplus$ {3, 4, 5} = {1, 2, 4, 5}
  - {New York, Washington} $\oplus$ {3, 4} = {New York, Washington, 3, 4}
  - {1, 2} $\oplus$ $\varnothing$ = {1, 2}
  - The symmetric difference of any set S with $\varnothing$ is S

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