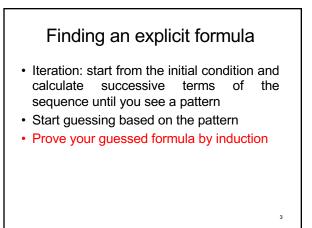


- An explicit formula for a recurrence relation is called a solution
- Given a sequence $a_0, a_1, a_2, ..., a_n$ defined by a recurrence relation, an explicit formula states a_n in terms of n only, without involving any previous terms

$$a_n = \sum_{i=1}^n i \Leftrightarrow a_n = \frac{n(n+1)}{2}$$

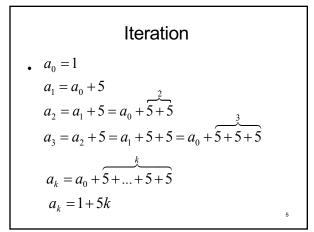


Arithmetic sequence

- Each term is the sum of the previous term and a constant: $a_k = a_{k-1} + d$
- Consider

$$a_0 = 1$$

 $a_k = a_{k-1} + 5$



Arithmetic Sequence

- $a_k = a_{k-1} + d$
 - $a_0 = x$
- $a_n = x + dn$

Geometric sequence

• Each term is the product of the previous term and a constant:

$$a_0 = x$$
$$a_k = ra_{k-1}$$

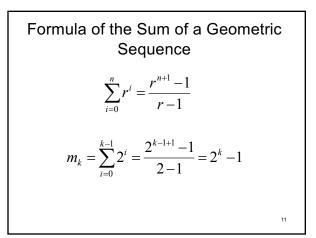
Explicit formula of a geometric
sequence
•
$$a_0 = x$$

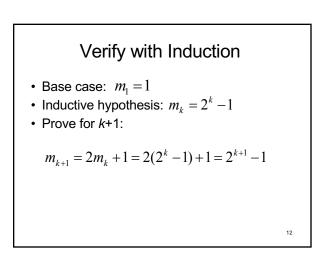
 $a_1 = ra_0$
 $a_2 = ra_1 = r^2 a_0$
 $a_3 = ra_2 = r^2 a_1 = r^3 a_0$
 $a_k = r^k a_0$
 $a_k = r^k x$

Growth of a Geometric Sequence	
10 ⁷	Number of seconds in a year
10 ⁹	Number of bytes of RAM in PC
1011	Number of neurons in a human brain
10 ¹⁷	Age of the universe in seconds
10 ³¹	Number of seconds to process all possible positions of a checkers game, process rate of 1 move per nano second
10^{81}	Number of atoms in the universe
10111	Number of seconds to process all possible positions of a chess game

Tower of Hanoi Sequence
Recall that the ToH sequence satisfies the
recurrence relation
$$m_k = 2m_{k-1} + 1, k \ge 2$$

 $m_1 = 1$
 $m_2 = 2m_1 + 1 = 2 + 1$
 $m_3 = 2m_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1 = 2^2 + 2^1 + 2^0$
 $m_4 = 2m_3 + 1 = 2(2^2 + 2^1 + 2^0) + 1 = 2^3 + 2^2 + 2^1 + 2^0$
 $m_k = 2^{k-1} + ... + 2^1 + 2^0 = \sum_{i=0}^{k-1} 2^i$

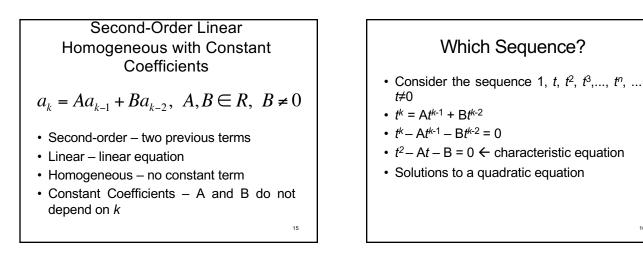




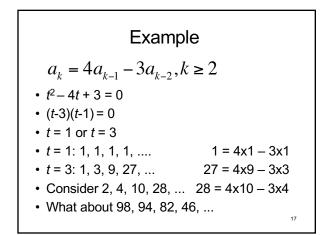
Example

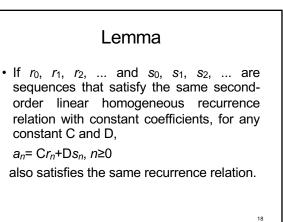
- $a_n = a_{n-1} + 2n 1$, $a_0 = 0$ Iteration: $-a_1 = 0 + 2x1 - 1 = 1$ $-a_2 = 1 + 2x^2 - 1 = 4$ $-a_3 = 4 + 2x3 - 1 = 9$
- $-a_4 = 9 + 2x4 1 = 16$
- $-a_5 = 16 + 2x5 1 = 25$
- Guess: *a_n* = *n*²

Verify with Induction
•
$$a_n = a_{n-1} + 2n - 1$$
, $a_0 = 0$
• Base case: $a_0 = 0^2 = 0$
• Inductive Hypothesis: $a_k = k^2$
• Inductive Step:
 $-a_{k+1} = a_k + 2(k+1) - 1$
 $-a_{k+1} = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$



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Proof

- *r*₀, *r*₁, *r*₂, ... and *s*₀, *s*₁, *s*₂, ... are sequences that satisfy the same SLHRRwCC
- $r_k = Ar_{k-1} + Br_{k-2}$ and $s_k = As_{k-1} + Bs_{k-2}$, $k \ge 2$
- $a_n = Cr_n + Ds_n$
- $a_n = C(Ar_{n-1} + Br_{n-2}) + D(As_{n-1} + Bs_{n-2})$
- $a_n = A(Cr_{n-1} + Ds_{n-1}) + B(Cr_{n-2} + Ds_{n-2})$
- *a*_{*n*} = A*a*_{*n*-1} + B*a*_{*n*-2} ■

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Distinct Root Theorem

• If *r* and *s* are distinct roots to the characteristic equation $t^2 - At - B = 0$ of a recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, $k \ge 2$, then the sequence is defined by the explicit formula:

 $a_n = \mathbf{C}r^n + \mathbf{D}s^n$,

where C and D are constants determined by a_0 and a_1 , if given.

• $a_0 = C + D$, $a_1 = Cr + Ds$

