Sequences

CS 231

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Definitions

- Sequence: an ordered list of elements
- A sequence is a function whose domain is a subset of $\mathcal{Z}$
  - Usually from the positive or non-negative integers
  - can be infinite
- a_n is a term in the sequence
- $\{a_n\}$ means the entire sequence

Sequence Examples

- $a_n = 3n$
  - The terms in the sequence are $a_1, a_2, a_3, \ldots$
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
- $b_n = 2^n$
  - The terms in the sequence are $b_1, b_2, b_3, \ldots$
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
- Sequences are indexed from 1
  - Not in all textbooks, though!

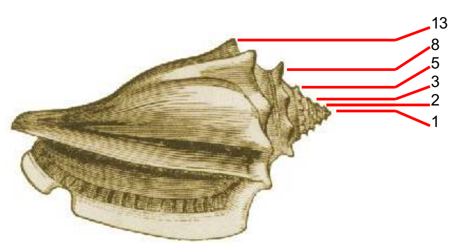
Geometric vs. Arithmetic Sequences

- The difference is in how they grow
- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$: { 3, 6, 9, 12, … }
  - Each number is 3 more than the previous
  - Of the form: $f(x) = dx + a$
- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$: { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

Fibonacci sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5} \cdot 2^n}$

Fibonacci sequence in nature



Reproducing rabbits

- You have one pair of rabbits on an island
  - The rabbits repeat the following:
    - Get pregnant one month
    - Give birth (to another pair) the next month
  - This process repeats indefinitely (no deaths)
  - Rabbits get pregnant the month they are born
- How many rabbits are there after 10 months?

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Reproducing rabbits

- First month: 1 pair
  - The original pair
- Second month: 1 pair
  - The original (and now pregnant) pair
- Third month: 2 pairs
  - The child pair (which is pregnant) and the parent pair (recovering)
- Fourth month: 3 pairs
  - "Grandchildren": Children from the baby pair (now pregnant)
  - Child pair (recovering)
  - Parent pair (pregnant)
- Fifth month: 5 pairs
  - Both the grandchildren and the parents reproduced
  - 3 pairs are pregnant (child and the two new born rabbit pairs)

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Reproducing rabbits

- Note the sequence:

  { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }

- The Fibonacci sequence again

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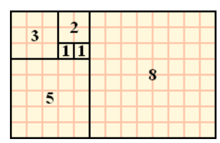
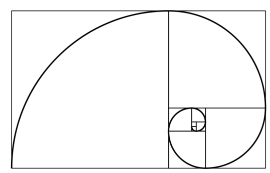
Fibonacci sequence

- Another application:





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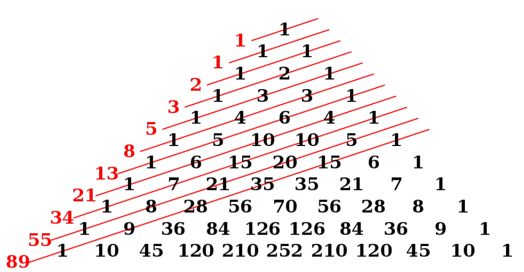
Pascal's Triangle



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Fibonacci sequence

- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)}=\varphi=\frac{\sqrt{5}+1}{2}=1.61803398874\ldots=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$$

- This is called the "golden ratio"
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

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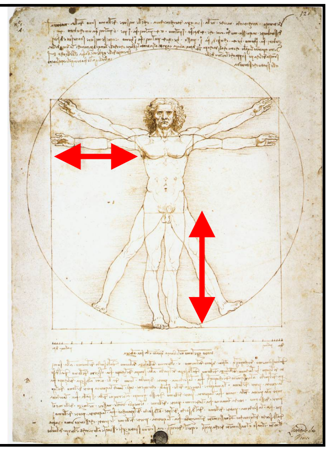
The Golden Ratio



Determining the sequence formula

- Given values in a sequence, how do you determine the explicit formula?
- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

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Determining the sequence formula

- 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
  - alternates 1's and 0's, increasing the number of 1's and 0's each time
- 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
  - increases by one, but repeats all even numbers once
- 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
  - non-0 numbers are a geometric sequence (2^n) interspersed with zeros
- 3, 6, 12, 24, 48, 96, 192, …
  - Each term is twice the previous: geometric progression
  - $a_n = 3*2^{n-1}$

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Determining the sequence formula

- 15, 8, 1, -6, -13, -20, -27, …
  - Each term is 7 less than the previous term
  - $a_n = 22 - 7n$
- 3, 5, 8, 12, 17, 23, 30, 38, 47, …
  - The difference between successive terms increases by one each time: $a_1 = 3$, $a_n = a_{n-1} + n$
  - $a_n = n(n+1)/2 + 2$
- 2, 16, 54, 128, 250, 432, 686, …
  - Each term is twice the cube of n
  - $a_n = 2*n^3$
- 2, 3, 7, 25, 121, 721, 5041, 40321
  - Each successive term is about n times the previous
  - $a_n = n! + 1$

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Summations

- A summation:

$$\sum_{i=m}^{n} a_i \quad \text{or} \quad \sum\nolimits_{i=m}^{n} a_i$$

upper limit

lower limit

index of summation

- is like a for loop:

```
sum:=0
for (i:= m to n)
    sum:=sum + ai
next i
```

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Evaluating sequences

$\sum_{k=1}^{5}(k+1)$
- 2 + 3 + 4 + 5 + 6 = 20

$\sum_{j=0}^{4}(-2)^j$
- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$\sum_{i=1}^{10}3$
- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$\sum_{j=0}^{8}\left(2^{j+1} - 2^j\right)$
- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^9-2^8) = 511$
  - Note that each term (except the first and last) is cancelled by another term

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More Notations

- Product: $\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots \times a_n$
- Factorial: $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$
- n choose r: $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$

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Properties

$$\sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} a_i + b_i$$

$$c \times \sum_{i=m}^{n} a_i = \sum_{i=m}^{n} c \times a_i$$

$$\prod_{i=m}^{n} a_i \times \prod_{i=m}^{n} b_i = \prod_{i=m}^{n} a_i \times b_i$$

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Double summations

- Like a nested for loop

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for (int i=1; i<=4; i++)
     for (int j=1; j<=3; j++)
          sum += i*j;
```

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