## Direct Proof and Counterexample

CS 231 Dianna Xu

#### Vacuous proofs

- Consider an implication:  $p \rightarrow q$
- If it can be shown that *p* is false, then the implication is always true
  - By definition of an implication/conditional
- Note that you are showing that the antecedent is false

#### Vacuous proof example

- · Consider the statement:
  - All criminology majors in CS 231 are female
     Rephrased: If you are a criminology major and you are in CS231, then you are female
- Since there are no criminology majors in this class, the antecedent is false, and the implication is true

#### Trivial proofs

• Consider an implication:  $p \rightarrow q$ 

consequent is true

- If it can be shown that q is true, then the implication is always true
   By definition of an implication
- Note that you are showing that the

### Trivial proof example

- Consider the statement:
   If you are in CS231 then you are human (domain is all people)
- Since all people are human, the implication is true regardless

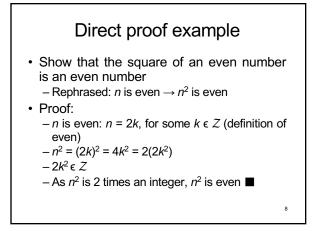
5

#### Direct proofs

- Consider an implication: p→q
   What if p is true, and q may or may not be
  - true? Show that if p is true, then q is true
- To perform a direct proof, assume that *p* is true, and show that *q* must be true

#### Definitions of Even and Odd

- *n* is even:  $\exists k \in \mathbb{Z}$  such that n = 2k
- *n* is odd:  $\exists k \in \mathbb{Z}$  such that n = 2k+1



#### **Proving Existential Statements**

- To prove a statement of the form  $-\exists x \in D, Q(x)$
- Constructive: find such an x
- · Non-constructive:
  - show that the existence of an x that makes Q(x) true is guaranteed by an axiom or a previously proved theorem – no need to find one
  - assume that there is no such x and show a contradiction

#### Constructive Existence Proof Example

- Show that a square exists that is the sum of two other squares
   Proof:3<sup>2</sup> + 4<sup>2</sup> = 5<sup>2</sup> ■
- Show that a cube exists that is the sum of three other cubes
   Proof: 3<sup>3</sup> + 4<sup>3</sup> + 5<sup>3</sup> = 6<sup>3</sup> ■
  - 10

# Non-constructive Existence Proof Prove that either 2x10<sup>500</sup>+15 or 2x10<sup>500</sup>+16 is not a perfect square A perfect square is the square of an integer Proof: The only two perfect squares that differ by 1 are 0 and 1 Thus, any other numbers that differ by 1 cannot both be perfect squares

- Thus, a non-perfect square must exist in any set that contains two numbers that differ by 1 ■
- Note that we didn't need to specify which one  $!_{\scriptscriptstyle 1}$

#### Disproving Universal Statements

- Disproving a statement of the form:  $\forall x \in D, P(x) \rightarrow Q(x)$
- Equivalent to showing the negation is true:
  - $-\exists x \in D, P(x) \text{ and } \sim Q(x)$
  - Finding such an *x* is known as finding a counterexample

12

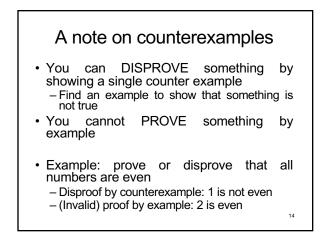
#### **Disproof by Counterexamples**

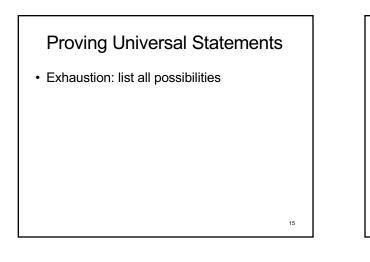
• Every positive integer is the square of another integer

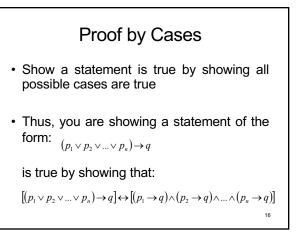
-√2 is not an integer ■

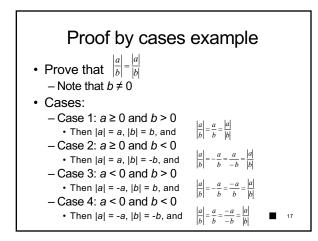
- If the sum of two integers is even, then one of them is even
   -1+3 = 4 ■

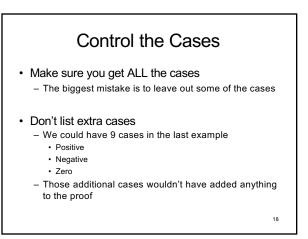
13







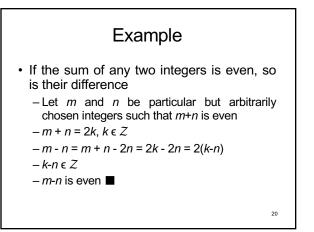


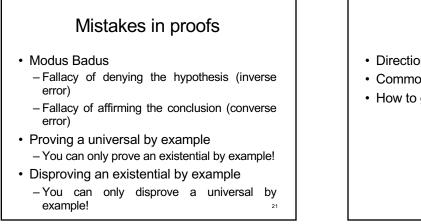


#### **Proving Universal Statements**

Generalizing from the generic particular

 show that every element of a set satisfies a certain property and that *x* is an element of such a set





19

#### Other Pointers

- Directions for writing Proofs
- Common mistakes
- · How to get a proof started

22