

Predicates and Quantifiers

CS 231
Dianna Xu

1

Predicates

- Consider $P(x) = x < 5$
 - $P(x)$ has no truth values (x is not given a value)
 - $P(1)$ is true – $1 < 5$ is true
 - $P(10)$ is false – $10 < 5$ is false
- Thus, $P(x)$ will become a statement/proposition when x is given a value

2

Truth set of predicates

- Let $P(x) = \text{“}x \text{ is a multiple of } 5\text{”}$
 - For what values of x is $P(x)$ true?
- Let $P(x) = x+1 > x$
 - For what values of x is $P(x)$ true?
- Let $P(x) = x + 3$
 - For what values of x is $P(x)$ true?

3

Multiple variables

- Functions/predicates with multiple variables:
 - $P(x,y) = x + y == 0$
 - $P(1,2)$ is false, $P(1,-1)$ is true
 - $P(x,y,z) = x + y == z$
 - $P(3,4,5)$ is false, $P(1,2,3)$ is true
 - $P(x_1, x_2, x_3 \dots x_n) = \dots$

4

Quantifiers

- A quantifier is “an operator that limits the variables of a predicate”
- Two types:
 - Universal
 - Existential

5

Universal quantifiers 1

- Represented by an upside-down A: \forall
 - It means “for all”
 - Let $P(x) = x+1 > x$
- We can state the following:
 - $\forall x, P(x)$
 - English translation: “for all values of x , $P(x)$ is true”
 - English translation: “for all values of x , $x+1 > x$ is true”

6

Universal quantifiers 2

- But is that always true?
 - $\forall x P(x)$
- Let x = the character 'a'
 - Is 'a'+1 > 'a'?
- Let x = the state of Pennsylvania
 - Is Pennsylvania+1 > Pennsylvania?
- You need to specify your universe!
 - What values x can represent
 - Known as the "domain" of x

7

Universal quantifiers 3

- $\forall x \in \mathcal{R}, P(x)$
- Let $P(x) = x/2 < x$
 - $\forall x \in \mathcal{R}, P(x)$?
- To prove that a universal quantification is true, it must be shown for ALL cases – **exhaustion**
- To prove that a universal quantification is false, it must be shown to be false for only ONE case – **counter example**

8

Existential quantification 1

- Represented by an backwards E: \exists
 - It means "there exists"
 - Let $P(x) = x+1 > x$
- We can state the following:
 - $\exists x, P(x)$
 - English translation: "there exists (a value of) x such that $P(x)$ is true"
 - English translation: "for at least one value of x , $x+1 > x$ is true"

9

Existential quantification 2

- Note that you still have to specify your domain
 - If the domain we are talking about is all the states in the US, then $\exists x P(x)$ is not true

10

Existential quantification 3

- In order to show an existential quantification is true, you only have to find ONE value
- In order to show an existential quantification is false, you have to show it's false for ALL values

11

A note on quantifiers

- $P(x) = x < 1$
- There are two ways to make a predicate (propositional function) into a statement (proposition):
 - Supply it with a value
 - For example, $P(5)$ is false, $P(0)$ is true
 - Provide a quantification
 - $\forall x \in \mathcal{Z}, P(x)$ is false and $\exists x \in \mathcal{Z}, P(x)$ is true

12

Universal Conditional Statements

- $\forall x$, if $P(x)$ then $Q(x)$
- $\forall x, P(x) \rightarrow Q(x)$
- $\forall x \in \mathcal{R}$, if $x > 0$ then $x+2 > 0$
- Consider “If a number is positive, then it is not zero”
- Implicit quantification
 - $P(x)$ implies $Q(x) \equiv \forall x, P(x) \rightarrow Q(x)$
 - $P(x)$ iff $Q(x) \equiv \forall x, P(x) \leftrightarrow Q(x)$

13

Translation from English

- Consider “All cats are black”
 - Let C be the set of all cats
 - $\forall x \in C$, x is black
 - Let $B(x)$ be “ x is black”: $\forall x \in C, B(x)$
- “Some people are crazy”
 - Let P be the set of all people
 - $\exists x \in P$, x is crazy
 - Let $\text{Crazy}(x)$ be “ x is crazy”: $\exists x \in P, \text{Crazy}(x)$

14

Translation from English

- Consider “Every student in this class has studied compound statements”
- Rephrased: “For every student x in this class, x has studied compound statements”
 - Let $C(x)$ be “ x has studied compound statements”
 - $\forall x, C(x)$, where the domain is “all students in this class”

15

Equivalent forms

- What if the domain is all people?
 - Let $S(x)$ be “ x is a student in this class”
 - Consider: $\forall x (S(x) \wedge C(x))$
- $\forall x, S(x) \rightarrow C(x)$
- $\forall x, C(x)$, where the domain is “all students in this class”
- $\forall x \in U, S(x) \rightarrow C(x) \equiv \forall x \in D, C(x)$, where D is the domain consisting of the truth set of $S(x)$

16

Translating from English

- Consider:
 - “Some students have visited Mexico”
 - “Every student has visited Canada or Mexico”
- Let:
 - $S(x)$ be “ x is a student”
 - $M(x)$ be “ x has visited Mexico”
 - $C(x)$ be “ x has visited Canada”

17

Translating from English

- Consider: “Some students have visited Mexico”
- $\exists x \in D, M(x)$ where $D =$ all students
- What if the domain is all people?
 - Consider: $\exists x, (S(x) \rightarrow M(x))$
 - $\exists x, (S(x) \wedge M(x))$

18

Translating from English

- Consider: “ Every student has visited Canada or Mexico”
- $\forall x \in D, M(x) \vee C(x)$
 - D = all students
- $\forall x, (S(x) \rightarrow (M(x) \vee C(x)))$

19

Negating quantifications

- Consider the statement:
 - All students in this class have red hair
- What is required to show the statement is false?
 - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
 - Negate the predicate
 - AND change to an existential quantification
 - Change “All are” to “Some are not”
 - $\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$

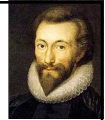
20

Negating quantifications 2

- Consider the statement:
 - There is a student in this class with red hair
- What is required to show the statement is false?
 - All students in this class do not have red hair
- Thus, to negate an existential quantification:
 - Negate the predicate
 - AND change to a universal quantification
 - $\sim(\exists x \in D, P(x)) \equiv \forall x \in D, \sim P(x)$

21

More negations



- “No man is an island” – John Donne
 - $\forall x, x$ is not an island
 - $\exists x, x$ is an island
 - Some men are islands
- “All that glitters is not gold” – Chaucer and Shakespeare

22

Negation of universal conditionals

- $\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D, \sim(P(x) \rightarrow Q(x))$
- $\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D, P(x) \wedge \sim Q(x)$
- For all people x, if x is rich then x is happy
 - There is one person who is rich and is not happy
 - There is one person who is rich but not happy

23

\forall and \wedge

- Given a predicate P(x) and values in the domain $\{x_1, \dots, x_n\}$
- The universal quantification $\forall x P(x)$ implies:

$$P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

24

\exists and \vee

- Given a predicate $P(x)$ and values in the domain $\{x_1, \dots, x_n\}$
- The existential quantification $\exists x P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$$

25

Translating from English

- Translate the statements:
 - “All hummingbirds are richly colored”
 - “No large birds live on honey”
 - “Birds that do not live on honey are dull in color”
 - “Hummingbirds are small”
- Assign our predicates
 - Let $P(x)$ be “ x is a hummingbird”
 - Let $Q(x)$ be “ x is large”
 - Let $R(x)$ be “ x lives on honey”
 - Let $S(x)$ be “ x is richly colored”
- Let our domain be all birds

26

Translating from English

- “All hummingbirds are richly colored”
 - $\forall x, (P(x) \rightarrow S(x))$
- “No large birds live on honey”
 - $\sim \exists x, (Q(x) \wedge R(x))$
 - Alternatively: $\forall x, (\sim Q(x) \vee \sim R(x))$
- “Birds that do not live on honey are dull in color”
 - $\forall x, (\sim R(x) \rightarrow \sim S(x))$
- “Hummingbirds are small”
 - $\forall x, (P(x) \rightarrow \sim Q(x))$

27