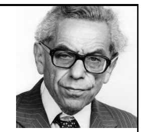
Graph Representation

CS231

Dianna Xu

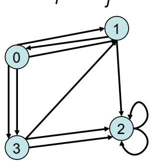
The Erdös Number



- Collaboration Graph
- Paul Erdös (1913-1996)
  - A prolific Hungarian mathematician
- E(Einstein) = 2, E(Turing) = 5, E(Nash) = 4
- Bacon number
- Erdös-Bacon number

Adjacency Matrix

- Given $G = (V, E)$ where $|V| = n$, the adjacency matrix A_G (A) of G is the nxn matrix where A_{ij} is the number of edges from v_i to v_j.



$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

Variations

- If G is undirected, then A_{ij} is the number of edges between v_i and v_j.
- The resulting A is symmetric.
- If G is a simple graph, then A_{ij} is binary.
- A is dependent on the ordering of V.
- How many different adjacency matrices represent the same graph?

Connected Components

- Let G be a graph with connected components G_1, …, G_k. Let n_i be the number of vertices in G_i. The adjacency matrix of G has the form:

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

Matrix Multiplication

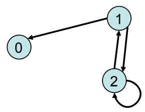
- Given matrices A and B, the product M = AB is defined as follows:

$$M_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

- Matrix multiplication does NOT commute.

Matrix Power

- Given a square matrix A, the powers of A are defined as follows:
  - $A^0 = I$
  - $A^n = AA^{n-1}$



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Theorem

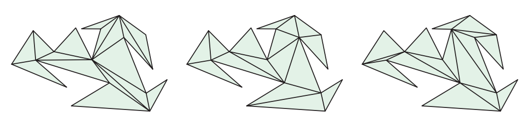
- Given $G = (V, E)$ with adjacency matrix A, the number of walks of length k from v_i to v_j is given by $(A^k)_{ij}$.
- Proof by induction:
  - P(1): A_{ij} = # of edges from v_i to v_j = # of walks of length 1 from v_i to v_j
  - Assume P(k): $(A^k)_{ij}$ = # of walks of length k from v_i to v_j
  - Prove P(k+1)

Proof

- P(k+1):
  - $A^{k+1} = AA^k$
  - $(A^{k+1})_{ij} = a_{i1}(A^k)_{1j} + a_{i2}(A^k)_{2j} + \ldots + a_{in}(A^k)_{nj}$
  - Consider $a_{i1}(A^k)_{1j}$:
    - By the inductive hypothesis, it is the # of walks of length k from v_1 to v_j multiplied by the # of walks of length 1 from v_i to v_1.
    - Which is the # of walks of length k+1 from v_i to v_j passing through v_1.
  - Argument holds for all terms
  - Thus the total is the number of all possible walks from v_i to v_j.

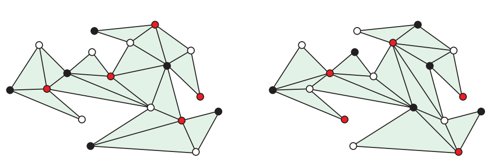
Triangulation

- A triangulation of a polygon is a decomposition into triangles with maximal non-crossing diagonals.

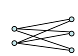


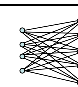
Graph Coloring

- A coloring of a graph is an assignment of colors to nodes so that no adjacent nodes have the same color



Bipartite Graphs

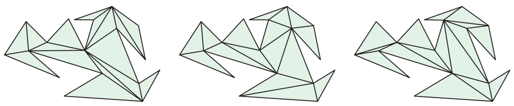




- A simple graph is *bipartite* if V can be partitioned into $V = V_1 \cup V_2$ so that any two adjacent vertices are in different partitions.
- A bipartite graph is *bichromatic (can be two-colored)*
  - vertices can be colored using two colors so that no two vertices of the same color are adjacent.

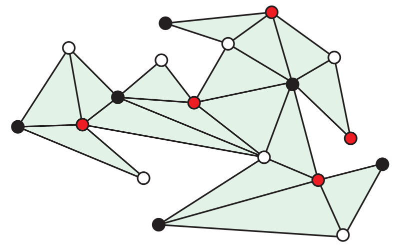
Triangulation of a Polygon

- A triangulation of a polygon is a decomposition into triangles with maximal non-crossing diagonals.



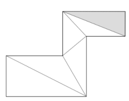
- A polygon is a simple circuit.
- A triangulation is a maximal planar supergraph of a polygon.

Every Triangulation of a Polygon Can be 3-colored



Meister's Two Ears

- Three consecutive vertices *a*, *b* and *c* on the boundary of a polygon form an ear if *ac* is a diagonal. *b* is known as an ear tip.



- Every polygon with $n>3$ vertices has at least two ears.