

Combinations

CS231
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Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
 - A♦, 5♥, 7♣, 10♠, K♣
- Is that the same hand as:
 - K♣, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
 - If yes, then we are dealing with permutations
 - If no, then we are dealing with combinations

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Permutations

- An r -permutation is an ordered arrangement of r elements of the set
 - A♦, 5♥, 7♣, 10♠, K♣ is a 5-permutation of the set of cards
- The notation for the number of r -permutations: $P(n,r)$

$$P(n,r) = \frac{n!}{(n-r)!}$$

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Combinations

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
 - A♦, 5♥, 7♣, 10♠, K♣
 - K♣, 10♠, 7♣, 5♥, A♦
- The number of r -combinations of a set with n elements, where n is non-negative and $0 \leq r \leq n$ is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

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Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

- How many different (initial) blackjack hands are there?

$$C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52 \times 51}{2 \times 1} = 1,326$$

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Combination formula proof

- Let $C(52,5)$ be the number of ways to generate unordered poker hands
- The number of ordered poker hands is $P(52,5) = 311,875,200$
- The number of ways to order a single poker hand is $P(5,5) = 5! = 120$
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, $C(52,5) = P(52,5)/P(5,5)$

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Combination formula proof

- Let $C(n,r)$ be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. r -permutations) is $P(n,r)$
- The number of ways to order a single one of those r -permutations $P(r,r)$
- The total number of unordered combinations is the total number of ordered combinations (i.e. r -permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

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Combination Formula

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

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Bit Strings

- How many bit strings of length 10 contain:
- Exactly four 1's?
 - Find the positions of the four 1's
 - Does the order of these positions matter?
 - Thus, the answer is $C(10,4) = 210$
- At most four 1's?
 - There can be 0, 1, 2, 3, or 4 occurrences of 1
 - $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)$
 - = $1+10+45+120+210$
 - = 386

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Bit Strings

- How many bit strings of length 10 contain:
- At least four 1's?
 - There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
 - $C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10)$
 - = $210+252+210+120+45+10+1$
 - = 848
 - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1
- An equal number of 1's and 0's?
 - Thus, there must be five 0's and five 1's
 - Find the positions of the five 1's
 - Thus, the answer is $C(10,5) = 252$

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Corollary 1

- Let n and r be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

- Proof:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

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Corollary example

- There are $C(52,5)$ ways to pick a 5-card poker hand
- There are $C(52,47)$ ways to pick a 47-card hand
- $P(52,5) = 2,598,960 = P(52,47)$
- When dealing 47 cards, you are picking 5 cards to not deal
 - As opposed to picking 5 card to deal
 - Again, the order the cards are dealt in does matter

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Note

- An alternative (and more common) way to denote an r -combination:

$$C(n, r) = \binom{n}{r}$$

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Choosing Teams

- Choosing team of 5 among 12
- Two members must work as a pair
 - # of teams that contain both: $C(10, 3) = 120$
 - # of teams that don't: $C(10, 5) = 252$
 - addition rule
- Two members must be kept apart
 - # of teams that have either: $2 \times C(10, 4) = 420$
 - # of teams that don't: $C(10, 5) = 252$

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Choosing Teams

- We have 5 men and 7 women
- How many 5-person groups can be chosen that
 - consist of 3 men and 2 women?
 - $C(5, 3) \times C(7, 2) = 210$
 - have at least one man?
 - $C(12, 5) - C(7, 5) = 771$
 - at most one man?
 - $C(7, 5) + C(5, 1) \times C(7, 4) = 196$

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r -Combinations with Repetitions

- How many 2-combinations can be selected from $\{1, 2, 3\}$, if repetitions are allowed?
 - $\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}$

1	2	3	selection	string
XX			{1,1}	xx
X	X		{1,2}	x x
X		X	{1,3}	x x
	XX		{2,2}	xx
	X	X	{2,3}	x x
		XX	{3,3}	xx

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r -Combinations with Repetitions

- xx||, x|x|, x||x, |xx|, |x|x, ||xx
- Strings of 4 symbols with 2 x's and 2 |'s
- Notice that once the positions of the x's are fixed, the |'s just go between
- $C(4, 2) = 4 \times 3 / 2 = 6$

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r -Combinations with Repetitions

- The number of r -Combinations with repetition allowed that can be selected from a set of n elements is: $C(r+n-1, r)$

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Soda Distribution

- Select 15 cans of soft drinks from 5 types
 - How many different selections?
 - $C(5+15-1, 15) = C(19, 15) = 3,876$
 - If Diet Coke is one of the types, how many selections include at least 6 cans Diet Coke?
 - choose the DCs first, then the rest
 - $C(5+9-1, 9) = C(13, 9) = 715$
 - If the store only has 5 cans of DC, but at least 15 cans of all others, how many selections?

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Ways to Count

- Choosing k elements from n

	order matters	order doesn't matter
Repetition allowed	n^k	$C(k+n-1, k)$
No repetition	$P(n, k)$	$C(n, k)$

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Circular seatings

- How many ways are there to seat 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?
 - First, place the first person in the north-most chair
 - Only one possibility
 - Then place the other 5 people
 - There are $P(5,5) = 5! = 120$ ways to do that
 - By the product rule, we get $1 \cdot 120 = 120$
 - Alternative means to answer this:
 - There are $P(6,6) = 720$ ways to seat the 6 people around the table
 - For each seating, there are 6 "rotations" of the seating
 - Thus, the final answer is $720/6 = 120$

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Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
 - Note that order does matter!
- Solution by cases
 - No ties
 - The number of permutations is $P(4,4) = 4! = 24$
 - Two horses tie
 - There are $C(4,2) = 6$ ways to choose the two horses that tie
 - There are $P(3,3) = 6$ ways for the "groups" to finish
 - A "group" is either a single horse or the two tying horses
 - By the product rule, there are $6 \cdot 6 = 36$ possibilities for this case
 - Two groups of two horses tie
 - There are $C(4,2) = 6$ ways to choose the two winning horses
 - The other two horses tie for second place
 - Three horses tie with each other
 - There are $C(4,3) = 4$ ways to choose the three horses that tie
 - There are $P(2,2) = 2$ ways for the "groups" to finish
 - By the product rule, there are $4 \cdot 2 = 8$ possibilities for this case
 - All four horses tie
 - There is only one combination for this
 - By the sum rule, the total is $24+36+8+1 = 75$

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Counting Triples

- How many (i, j, k) such that $1 \leq i \leq j \leq k \leq n$?
- If $n=5$, represent $(3, 3, 4)$ as $||xx|x|$
- If $n=7$, represent $(2, 4, 5)$ as $|x||x|x||$
- How many 's'?
- How many x's?
- $C(3+n-1, 3) = (n+2)!/3!(n-1)!$
 $= (n+2)(n+1)n/6$

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Nested for loop

- ```

for (k:= 1 to n)
 for (j:= 1 to k)
 for (i:= 1 to j)
 //body
 next i
 next j
next k

```
- How many times will the innermost loop body be executed?
  - For each iteration, there is a different combination of the indices  $(i, j, k)$ ,  $1 \leq i \leq j \leq k \leq n$

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