The Pigeonhole Principle

CS231
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Too many pigeons

The pigeonhole principle

- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it
- A function from one finite set to a smaller finite set can not be one-to one: there must be at least two elements in the domain that have the same image in the co-domain

Pigeonhole principle examples

- In a group of 367 people, there must be two people with the same birthday
  - As there are 366 possible birthdays
- In a group of 27 English words, at least two words must start with the same letter
  - As there are only 26 letters

Hair Count

- Among the residents of Philadelphia, there must be at least two people with the same number of hairs on their heads
- Pigeons
  - population of Philadelphia
  - > 1.5 million
- Holes:
  - # of hairs on human head
  - < 300,000

Generalized pigeonhole principle

- For any function f from a finite set X to a finite set Y and for any positive integer k, if $|X| > k|Y|$, then there is some $y \in Y$ such that y is the image of at least k+1 distinct elements of X

$k+1 = \lceil |X|/|Y| \rceil$

Equivalent Statements

- If *m* compartments contain *km*+1 objects, then at least one compartment contains *k* +1 or more objects
- If all *m* compartments contain at most *k* elements, then there can not be more than *km* elements.
- For a non-empty, finite bag of numbers, the maximum value is at least the average value.

Generalized pigeonhole principle examples

- Among 100 people, there are at least $\lceil 100/12 \rceil = 9$ born on the same month
- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?
  - The "holes" are the grades. Thus, $k = 5$
  - Thus, we set $\lceil N/5 \rceil = 6$
  - Lowest possible value for N is 26

Sample questions

- A bowl contains 10 red and 10 yellow balls: how many balls must be selected to ensure 3 balls of the same color?
  - Consider the "worst" case
    - You have 2 balls of each color
    - You can't take another ball without hitting 3
    - Thus, the answer is 5

Sample questions

- Via generalized pigeonhole principle
  - How many balls are required if there are 2 colors, and one color must have 3 balls?
  - Number of pigeon holes: $k = 2$
  - Min number of pigeons in one hole: $\lceil N/k \rceil = 3$
  - Solve for N: $N = 5$

Sample questions

- How many balls must be selected to ensure 3 yellow balls?
  - Consider the "worst" case
    - Consider 10 red balls and 2 yellow balls
    - You can't take another ball without hitting 3 yellow balls
    - Thus, the answer is 13

Sample questions

- 6 computers on a network are connected to at least 1 other computer
- Show there are at least two computers that are have the same number of connections

Sample questions

- The number of holes, *k*, is the number of computer connections
  - 1, 2, 3, 4 or 5
- The number of pigeons, *N*, is the number of computers
  - 6
- By the generalized pigeonhole principle, at least one box must have $\lceil N/k \rceil$ objects
  - $\lceil 6/5 \rceil = 2$
  - In other words, at least two computers must have the same number of connections

Friends

- In any group of people (>1), there must be at least two people who have the same number of friends
  1. Everyone has at least one friend
  2. Someone has no friends

Sample question

- Consider 5 distinct points (x_i, y_i) with integer coordinate values
- Show that the midpoint of at least one pair of these five points also has integer coordinates

Sample question

- We are looking for the midpoint of a segment from (a,b) to (c,d): ((a+c)/2, (b+d)/2)
- The coordinates will be integers if a and c (resp. b and d) have the same parity: are either both even or both odd
- There are four parity possibilities
  - (even, even), (even, odd), (odd, even), (odd, odd)
- Since we have 5 points, by the pigeonhole principle, there must be two points that have the same parity possibility