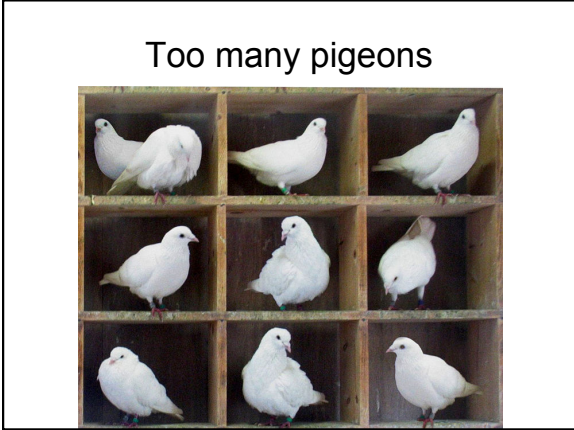


# The Pigeonhole Principle

CS231  
Dianna Xu



## The pigeonhole principle

- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it
- A function from one finite set to a smaller finite set can not be one-to one: there must be at least two elements in the domain that have the same image in the co-domain

## Pigeonhole principle examples

- In a group of 367 people, there must be two people with the same birthday
  - As there are 366 possible birthdays
- In a group of 27 English words, at least two words must start with the same letter
  - As there are only 26 letters

## Hair Count

- Among the residents of Philadelphia, there must be at least two people with the same number of hairs on their heads
- Pigeons
  - population of Philadelphia
  - > 1.5 million
- Holes:
  - # of hairs on human head
  - < 300,000

## Generalized pigeonhole principle

- For any function  $f$  from a finite set  $X$  to a finite set  $Y$  and for any positive integer  $k$ , if  $|X| > k|Y|$ , then there is some  $y \in Y$  such that  $y$  is the image of at least  $k+1$  distinct elements of  $X$

$$k+1 = \lceil |X|/|Y| \rceil$$

### Equivalent Statements

- If  $m$  compartments contain  $km+1$  objects, then at least one compartment contains  $k+1$  or more objects
- If all  $m$  compartments contain at most  $k$  elements, then there can not be more than  $km$  elements.
- For a non-empty, finite bag of numbers, the maximum value is at least the average value.

### Generalized pigeonhole principle examples

- Among 100 people, there are at least  $\lceil 100/12 \rceil = 9$  born on the same month
- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?
  - The “holes” are the grades. Thus,  $k = 5$
  - Thus, we set  $\lceil N/5 \rceil = 6$
  - Lowest possible value for  $N$  is 26

### Sample questions

- A bowl contains 10 red and 10 yellow balls: how many balls must be selected to ensure 3 balls of the same color?
  - Consider the “worst” case
    - You have 2 balls of each color
    - You can’t take another ball without hitting 3
    - Thus, the answer is 5

### Sample questions

- Via generalized pigeonhole principle
  - How many balls are required if there are 2 colors, and one color must have 3 balls?
  - Number of pigeon holes:  $k = 2$
  - Min number of pigeons in one hole:  $\lceil N/k \rceil = 3$
  - Solve for  $N$ :  $N = 5$

### Sample questions

- How many balls must be selected to ensure 3 yellow balls?
  - Consider the “worst” case
    - Consider 10 red balls and 2 yellow balls
    - You can’t take another ball without hitting 3 yellow balls
    - Thus, the answer is 13

### Sample questions

- 6 computers on a network are connected to at least 1 other computer
- Show there are at least two computers that are have the same number of connections

### Sample questions

- The number of holes,  $k$ , is the number of computer connections
  - 1, 2, 3, 4 or 5
- The number of pigeons,  $N$ , is the number of computers
  - 6
- By the generalized pigeonhole principle, at least one box must have  $\lceil N/k \rceil$  objects
  - $\lceil 6/5 \rceil = 2$
  - In other words, at least two computers must have the same number of connections

### Friends

- In any group of people ( $>1$ ), there must be at least two people who have the same number of friends
  1. Everyone has at least one friend
  2. Someone has no friends

### Sample question

- Consider 5 distinct points  $(x_i, y_i)$  with integer coordinate values
- Show that the midpoint of at least one pair of these five points also has integer coordinates

### Sample question

- We are looking for the midpoint of a segment from  $(a,b)$  to  $(c,d)$ :  $((a+c)/2, (b+d)/2)$
- The coordinates will be integers if  $a$  and  $c$  (resp.  $b$  and  $d$ ) have the same parity: are either both even or both odd
- There are four parity possibilities
  - (even, even), (even, odd), (odd, even), (odd, odd)
- Since we have 5 points, by the pigeonhole principle, there must be two points that have the same parity possibility