The Addition Rule

CS231

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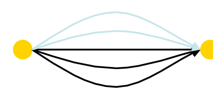
The addition rule

- If a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$,
- $|A| = |A_1| + |A_2| + \ldots + |A_k|$
- If there are n_1 ways to do task 1, and n_2 ways to do task 2
  - Then there are n_1+n_2 ways to do one of the two tasks
  - We must make one choice OR a second choice



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Sum rule example

- Sample question
  - There are 18 math majors and 17 CS majors
  - How many ways are there to pick one math major or one CS major?
- Total is 18 + 17 = 35

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Sum rule example

More sample questions

- How many strings of 4 decimal digits…
- Have exactly three digits that are 9s?
  - The string can have:
    - The non-9 as the first digit
    - OR the non-9 as the second digit
    - OR the non-9 as the third digit
    - OR the non-9 as the fourth digit
    - Thus, we use the sum rule
  - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
  - Thus, the answer is 9+9+9+9 = 36

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The difference rule

- If B is a subset of a finite set A
- |A-B| = |A| - |B|
- How many strings of 4-digits have repeated digits?
  - Total: 10x10x10x10 = 10000
  - No repeats: 10x9x8x7 = 5040
  - Repeats = 10000-5040 = 4960

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More complex counting problems

- We combining the product rule and the sum/difference rules
- Thus we can solve more interesting and complex problems

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Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom
- How many possibilities are there if the bride must be in the picture?
  - Product rule: place the bride AND then place the rest of the party
  - First place the bride
    - She can be in one of 6 positions
  - Next, place the other five people via the product rule
    - There are 9 people to choose for the second person, 8 for the third, etc.
    - Total = P(9,5) = 9x8x7x6x5 = 15120
  - Product rule yields 6 x P(9,5) = 90,720 possibilities

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Wedding pictures example

- How many possibilities are there if the bride and groom must both be in the picture?
  - Product rule: place the bride/groom AND then place the rest of the party
  - First place the bride and groom
    - She can be in one of 6 positions
    - He can be in one of 5 remaining positions
    - P(6,2) = 30 possibilities
  - Next, place the other four people via the product rule
    - There are 8 people to choose for the third person, 7 for the fourth, etc.
    - Total = P(8,4) = 8x7x6x5 = 1680
  - Product rule: P(6,2) x P(8,4) = 50,400 possibilities

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Wedding pictures example

- How many possibilities are there if only one of the bride and groom are in the picture
  - Sum rule: place only the bride
    - Product rule: place the bride AND then place the rest of the party
    - First place the bride
      - She can be in one of 6 positions
    - Next, place the other five people via the product rule
      - There are 8 people to choose for the second person, 7 for the third, etc.
        - » We can't choose the groom!
      - Total = P(8,5) = 8x7x6x5x4 = 6720
    - Product rule yields 6 x P(8,5) = 40,320 possibilities
  - OR place only the groom
    - Same possibilities as for bride: 40,320
  - Sum rule yields 2x(6xP(8,5)) = 80,640 possibilities

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Wedding pictures example

- Alternative means to get the answer
- Total ways to place the bride (with or without groom): 90,720
- Total ways for both the bride and groom: 50,400
- Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
- Same number for the groom
- Total = 40,320 + 40,320 = 80,640

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The inclusion-exclusion principle

- When counting the possibilities, we can't include a given outcome more than once!
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - Let A_1 have 5 elements, A_2 have 3 elements, and 1 element be both in A_1 and A_2
  - Total in the union is 5+3-1 = 7, not 8

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Inclusion-exclusion example

- How many bit strings of length eight start with 1 or end with 00?
- Count bit strings that start with 1
  - Rest of bits can be anything: $2^7 = 128 = |A_1|$
- Count bit strings that end with 00
  - Rest of bits can be anything: $2^6 = 64 = |A_2|$
- Count bit strings that both start with 1 and end with 00
  - Rest of the bits can be anything: $2^5 = 32 = |A_1 \cap A_2|$
- Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Total is 128 + 64 – 32 = 160

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Bit string possibilities

- How many bit strings of length 10 contain either (exactly) 5 consecutive 0s or (exactly) 5 consecutive 1s?
- Consider 5 consecutive 0s first.
- Sum rule: the 5 consecutive 0's can start at position 1, 2, 3, 4, 5, or 6

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Consecutive 0s

- Starting at position 1
  - 6th bit must be a 1 and last 4 bits can be anything: $2^4 = 16$
- Starting at position 2
  - First bit and 7th bit must be a 1
    - Otherwise, we are counting 6 consecutive 0s
  - Remaining bits can be anything: $2^3 = 8$
- Starting at position 3
  - Second and 8th bit must be a 1 (same reason as above)
  - First bit and last 2 bits can be anything: $2^3 = 8$
- Starting at positions 4 and 5
  - Same as starting at positions 2 or 3: 8 each
- Starting at position 6 – same as starting at 1 – 16
- Total = 16 + 8+ 8 + 8 + 8 + 16 = 64

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Double Counting?

- The 5 consecutive 1's follow the same pattern, and have 64 possibilities
- There are two cases counted twice (that we thus need to exclude): 0000011111 and 1111100000
- Total = 64 + 64 – 2 = 126

- How many bit strings of length 10 contain either (at least) 5 consecutive 0s or (at least) 5 consecutive 1s?

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