Sets

CS 231
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What is a set?

- A set is a group of "objects"
  - People: {Alice, Bob, Clara}
  - Colors of a rainbow: {red, orange, yellow, green, blue, purple}
  - States in the US: {Alabama, Alaska, Virginia, …}
  - All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}
  - Unrelated elements: {3, a, red, Virginia}

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Notes

- Order does not matter
  - We often write them in order because it is easier for humans to understand it that way
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}
- Sets are notated with curly brackets
- No duplicate elements
- Note that a list is like a set, but order does matter and duplicate elements are allowed

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Specifying a Set

- List all the elements: A = {1, 2, 3, 4, 5}
  - Not always feasible for large or infinite sets
- Set-builder notation
  - D = $\{x \mid x$ is prime and $x > 2\}$
  - all elements x such that x is prime and x is greater than 2
  - E = $\{x \in Z \mid x > 2\}$
  - all integers x such that x is greater than 2

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Specifying a Set

- A set is said to "contain" the various "members" or "elements" that make up the set
  - If an element *a* is a member of (or an element of) a set S, we use then notation $a \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
    - $7 \notin \{1, 2, 3, 4\}$
    - Virginia $\notin \{1, 2, 3, 4\}$

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The Universal Set

- ***U*** is the universal set – the set of all of elements (or the "universe") from which given any set is drawn
  - For the set {-2, 0.4, 2}, ***U*** would be the real numbers
  - For the set {0, 1, 2}, ***U*** could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context

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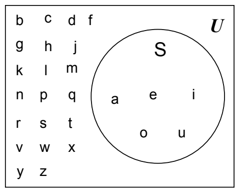
Venn diagrams

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram



Sets of sets

- Sets can contain other sets
  - S = { {1}, {2}, {3} }
  - T = { {1}, {{2}}, {{{3}}} }
  - V = { {{1}, {{2}}}, {{{3}}}, { {1}, {{2}}, {{{3}}} } }
    - V has only 3 elements!
- Note that 1 ≠ {1} ≠ {{1}} ≠ {{{1}}}
  - They are all different

The Empty Set

- If a set has zero elements, it is called the empty (or null) set
  - Written using the symbol ∅
  - Thus, ∅ = { }
- As the empty set is a set, it can be a element of other sets
  - { ∅, 1, 2, 3, x } is a valid set

The Empty Set

- Note that ∅ ≠ { ∅ }
  - The first is a set of zero elements
  - The second is a set of 1 element (that one element being the empty set)
- Replace ∅ by { }, and you get: { } ≠ { { } }
  - It's easier to see that they are not equal that way

Set Equality

- Two sets are equal iff they have the same elements
  - {1, 2, 3, 4, 5} = {5, 4, 3, 2, 1}
  - {1, 2, 3, 2, 4, 3, 2, 1} = {4, 3, 2, 1}
  - {1, 2, 3, 4, 5} ≠ {1, 2, 3, 4}

Subsets

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}
  - This is specified by S ⊆ T
- If S is not a subset of T, it is written as such:
  S ⊈ T
  - For example, {1, 2, 8} ⊈ {1, 2, 3, 4, 5, 6, 7}

Subsets

- Any set is a subset of itself!
  - Since all the elements of S are elements of S, $S \subseteq S$
- The empty set is a subset of *all* sets (including itself!)
- *All* sets are subsets of the universal set

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Proper Subsets

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - denoted as $S \subset T$
- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

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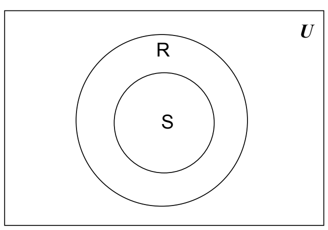
Proper Subsets: Venn Diagram

$S \subset R$



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Set Cardinality

- The cardinality of a set is the number of elements in a set
  - Written as |A|
- Examples
  - Let R = {1, 2, 3, 4, 5}. Then |R| = 5
  - |∅| = 0
  - Let S = {∅, {a}, {b}, {a, b}}. Then |S| = 4
- A set with one element is sometimes called a singleton set

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Power Sets

- Given the set S = {0, 1}. What are all the possible subsets of S?
  - They are: ∅, {0}, {1}, and {0, 1}
- The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
    - Note that |S| = 2 and |P(S)| = 4

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Power Sets

- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }
    - Note that |T| = 3 and |P(T)| = 8
- P(∅) = { ∅ }
    - Note that |∅| = 0 and |P(∅)| = 1
- If a set has *n* elements, then the power set will have 2^n elements

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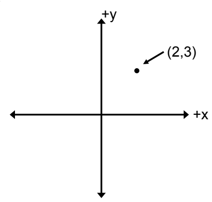
Tuples

- In 2-dimensional space, it is an **ordered** pair of numbers (for example, to specify a location)
- In *n*-dimensional space, it is a *n*-tuple of ordered numbers



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Cartesian Product

- A Cartesian product is a set of all ordered 2-tuples where each "part" is from a given set: A x B = { (a,b) | $a \in$ A and $b \in$ B }
- Given A = {a, b} and B = {0, 1}, what is their Cartesian product?
  - C = A x B = {(a,0), (a,1), (b,0), (b,1)}
- 2-D Cartesian coordinates are the set of all ordered pairs $\mathcal{Z}$x$\mathcal{Z}$ (or $\mathcal{R}$x$\mathcal{R}$)
- A 3-D coordinate is an element from the Cartesian product of $\mathcal{Z}$x$\mathcal{Z}$x$\mathcal{Z}$ (or $\mathcal{R}$x$\mathcal{R}$x$\mathcal{R}$)

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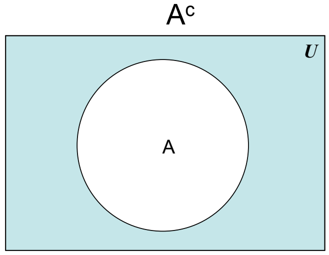
Complement

A^c



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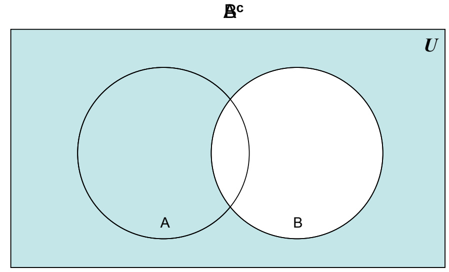
Complement

B^c



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Complement

- Complement of a set:
  $A^c = \{ x \mid x \notin A \}$
- Also written as $\bar{A}$
- Further examples (assuming $U = \mathcal{Z}$)
  - $\{1, 2, 3\}^c = \{ \ldots, -2, -1, 0, 4, 5, 6, \ldots \}$

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Complement

- Properties of complement sets
  - $(A^c)^c = A$ — Double Complement
  - $A \cup A^c = U$ — Complement law
  - $A \cap A^c = \varnothing$ — Complement law

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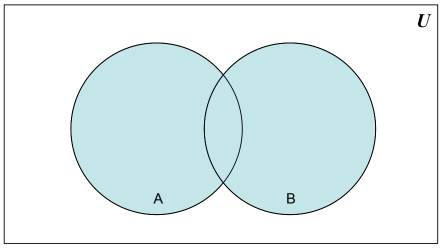
Union

A U B



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Union

- Union of two sets:
  A U B = { x | $x \in$ A $\vee$ $x \in$ B }
- Further examples
  - {1, 2, 3} U {3, 4, 5} = {1, 2, 3, 4, 5}
  - {New York, Washington} U {3, 4} = {New York, Washington, 3, 4}
  - {1, 2} U ∅ = {1, 2}

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Union Properties

- Properties of the union operation

– A U ∅ = A	Identity law
– A U ***U*** = ***U***	Universal Bound
– A U A = A	Idempotent law
– A U B = B U A	Commutative law
– A U (B U C) = (A U B) U C	Associative law

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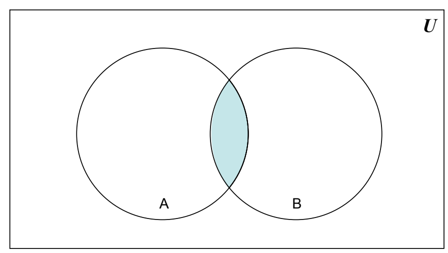
Intersection

A ∩ B



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Intersection

- Intersection of two sets:
  A ∩ B = { x | $x \in$ A $\wedge$ $x \in$ B }
- Further examples
  - {1, 2, 3} ∩ {3, 4, 5} = {3}
  - {New York, Washington} ∩ {3, 4} = ∅
    - No elements in common
  - {1, 2} ∩ ∅ = ∅
    - Any set intersection with the empty set yields the empty set

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Intersection Properties

- Properties of the intersection operation

– A ∩ ***U*** = A	Identity law
– A ∩ ∅ = ∅	Universal Bound
– A ∩ A = A	Idempotent law
– A ∩ B = B ∩ A	Commutative law
– A ∩ (B ∩ C) = (A ∩ B) ∩ C	Associative law

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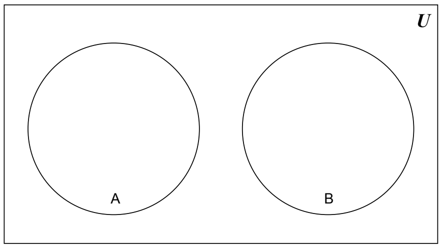
Disjoint Sets



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Disjoint sets

- Disjoint sets: two sets are disjoint if their intersection is the empty set
- Further examples
  - {1, 2, 3} and {3, 4, 5} are not disjoint
  - {New York, Washington} and {3, 4} are disjoint
  - {1, 2} and ∅ are disjoint
    - Their intersection is the empty set
  - ∅ and ∅ are disjoint!
    - Their intersection is the empty set

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Mutually Disjoint

- Sets A_1, A_2, …, A_n are mutually disjoint iff no two distinct sets A_i and A_j have any elements in common, in other words, $A_i \cap A_j = \varnothing$.
- A collection of nonempty sets {A_1, A_2, A_3 …} is as partition of a set $\mathcal{A}$ iff,
  - $\mathcal{A}$ is the union of all A_i
  - A_1, A_2, A_3, … are mutually disjoint

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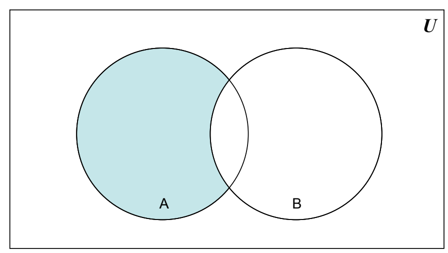
Difference

A - B



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Difference

- Difference of two sets:
  A - B = { x | $x \in A \wedge x \notin B$ }
  A - B = A ∩ B^c ← Important!
- Further examples
  - {1, 2, 3} - {3, 4, 5} = {1, 2}
  - {New York, Washington} - {3, 4} = {New York, Washington}
  - {1, 2} - ∅ = {1, 2}
  - The difference of any set S with ∅ is S

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Symmetric Difference

- Symmetric difference of two sets:
  A ⊕ B = { x | ($x \in A \vee x \in B$) ∧ $x \notin A \cap B$}
  A ⊕ B = (A U B) – (A ∩ B) ← Important!
- Further examples
  - {1, 2, 3} ⊕ {3, 4, 5} = {1, 2, 4, 5}
  - {New York, Washington} ⊕ {3, 4} = {New York, Washington, 3, 4}
  - {1, 2} ⊕ ∅ = {1, 2}
  - The symmetric difference of any set S with ∅ is S

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