Solving Recurrence Relations

CS231
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Explicit formula

- An explicit formula for a recurrence relation is called a solution
- Given a sequence $a_0, a_1, a_2, \ldots, a_n$ defined by a recurrence relation, an explicit formula states a_n in terms of n only, without involving any previous terms

$$a_n = \sum_{i=1}^{n} i \Leftrightarrow a_n = \frac{n(n+1)}{2}$$

Finding an explicit formula

- Iteration: start from the initial condition and calculate successive terms of the sequence until you see a pattern
- Start guessing based on the pattern
- Prove your guessed formula by induction

Arithmetic sequence

- Each term is the sum of the previous term and a constant: $a_k = a_{k-1} + d$
- Consider

$$a_0 = 1$$
$$a_k = a_{k-1} + 5$$

Iteration

- $a_0 = 1$

$$a_1 = a_0 + 5$$
$$a_2 = a_1 + 5 = a_0 + \overbrace{5 + 5}^{2}$$
$$a_3 = a_2 + 5 = a_1 + 5 + 5 = a_0 + \overbrace{5 + 5 + 5}^{3}$$
$$a_k = a_0 + \overbrace{5 + \ldots + 5 + 5}^{k}$$
$$a_k = 1 + 5k$$

Arithmetic Sequence

- $a_k = a_{k-1} + d$
  $a_0 = x$
- $a_n = x + dn$

Geometric sequence

- Each term is the product of the previous term and a constant:

$$a_0 = x$$
$$a_k = ra_{k-1}$$

7

Explicit formula of a geometric sequence

- $a_0 = x$

$$a_1 = ra_0$$
$$a_2 = ra_1 = r^2 a_0$$
$$a_3 = ra_2 = r^2 a_1 = r^3 a_0$$

$$a_k = r^k a_0$$
$$a_k = r^k x$$

8

Growth of a Geometric Sequence

10^7	**Number of seconds in a year**
10^9	**Number of bytes of RAM in PC**
10^{11}	**Number of neurons in a human brain**
10^{17}	**Age of the universe in seconds**
10^{31}	**Number of seconds to process all possible positions of a checkers game, process rate of 1 move per nano second**
10^{81}	**Number of atoms in the universe**
10^{111}	**Number of seconds to process all possible positions of a chess game**

9

Tower of Hanoi Sequence

Recall that the ToH sequence satisfies the recurrence relation $m_k = 2m_{k-1} + 1, k \geq 2$

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1 = 2^2 + 2^1 + 2^0$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2^1 + 2^0) + 1 = 2^3 + 2^2 + 2^1 + 2^0$$
$$m_k = 2^{k-1} + \ldots + 2^1 + 2^0 = \sum_{i=0}^{k-1} 2^i$$

10

Formula of the Sum of a Geometric Sequence

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

$$m_k = \sum_{i=0}^{k-1} 2^i = \frac{2^{k-1+1} - 1}{2 - 1} = 2^k - 1$$

11

Verify with Induction

- Base case: $m_1 = 1$
- Inductive hypothesis: $m_k = 2^k - 1$
- Prove for *k*+1:

$$m_{k+1} = 2m_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$$

12

Example

- $a_n = a_{n-1} + 2n - 1$, $a_0 = 0$
- Iteration:
  - $a_1 = 0 + 2x1 - 1 = 1$
  - $a_2 = 1 + 2x2 - 1 = 4$
  - $a_3 = 4 + 2x3 - 1 = 9$
  - $a_4 = 9 + 2x4 - 1 = 16$
  - $a_5 = 16 + 2x5 - 1 = 25$
- Guess: $a_n = n^2$

13

Verify with Induction

- $a_n = a_{n-1} + 2n - 1$, $a_0 = 0$
- Base case: $a_0 = 0^2 = 0$
- Inductive Hypothesis: $a_k = k^2$
- Inductive Step:
  - $a_{k+1} = a_k + 2(k+1) - 1$
  - $a_{k+1} = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$ ■

14

Second-Order Linear Homogeneous with Constant Coefficients

$$a_k = Aa_{k-1} + Ba_{k-2},\ A, B \in R,\ B \neq 0$$

- Second-order – two previous terms
- Linear – linear equation
- Homogeneous – no constant term
- Constant Coefficients – A and B do not depend on k

15

Which Sequence?

- Consider the sequence 1, t, t^2, t^3,..., t^n, ... $t \neq 0$
- $t^k = At^{k-1} + Bt^{k-2}$
- $t^k - At^{k-1} - Bt^{k-2} = 0$
- $t^2 - At - B = 0$ ← characteristic equation
- Solutions to a quadratic equation

16

Example

$$a_k = 4a_{k-1} - 3a_{k-2}, k \geq 2$$

- $t^2 - 4t + 3 = 0$
- $(t-3)(t-1) = 0$
- $t = 1$ or $t = 3$
- $t = 1$: 1, 1, 1, 1, ... $1 = 4x1 - 3x1$
- $t = 3$: 1, 3, 9, 27, ... $27 = 4x9 - 3x3$
- Consider 2, 4, 10, 28, ... $28 = 4x10 - 3x4$
- What about 98, 94, 82, 46, ...

17

Lemma

- If r_0, r_1, r_2, ... and s_0, s_1, s_2, ... are sequences that satisfy the same second-order linear homogeneous recurrence relation with constant coefficients, for any constant C and D,

  $a_n = Cr_n + Ds_n$, $n \geq 0$

also satisfies the same recurrence relation.

18

Proof

- $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same SLHRRwCC
- $r_k = Ar_{k-1} + Br_{k-2}$ and $s_k = As_{k-1} + Bs_{k-2}$, k≥2
- $a_n = Cr_n + Ds_n$
- $a_n = C(Ar_{n-1} + Br_{n-2}) + D(As_{n-1} + Bs_{n-2})$
- $a_n = A(Cr_{n-1} + Ds_{n-1}) + B(Cr_{n-2} + Ds_{n-2})$
- $a_n = Aa_{n-1} + Ba_{n-2}$ ■

19

Distinct Root Theorem

- If r and s are distinct roots to the characteristic equation $t^2 - At - B = 0$ of a recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, $k \geq 2$, then the sequence is defined by the explicit formula:

  $a_n = Cr^n + Ds^n$,

  where C and D are constants determined by a_0 and a_1, if given.
- $a_0 = C + D$, $a_1 = Cr + Ds$

20

Proof by Strong Induction

- Bases:
  - $a_0 = Cr^0 + Ds^0 = C + D$
  - $a_1 = Cr^1 + Ds^1 = Cr + Ds$
- Inductive Hypothesis: $a_k = Cr^k + Ds^k$, $k \geq 2$
- Inductive Step:
  - $a_{k+1} = Aa_k + Ba_{k-1} = A(Cr^k + Ds^k) + B(Cr^{k-1} + Ds^{k-1})$
  - $a_{k+1} = C(Ar^k + Br^{k-1}) + D(As^k + Bs^{k-1})$
  - $a_{k+1} = Cr^{k+1} + Ds^{k+1}$ ■

21

The Fibonacci Sequence

- $a_k = a_{k-1} + a_{k-2}$, $k \geq 2$, $a_0 = 1$, $a_1 = 1$
- Characteristic equation: $t^2 - t - 1 = 0$
- Roots: $a = 1$, $b = -1$, $c = -1$
  - $\Delta = (-1)^2 - 4 \times 1 \times (-1) = 5$
  - $x = (-(-1) \pm \sqrt{\Delta})/2$
  - $x_1 = (1+\sqrt{5})/2$
  - $x_2 = (1-\sqrt{5})/2$

22

Fibonacci Sequence Explicit Formula

$$a_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\left\{ \begin{array}{c} a_0 = 1 = C + D \\ a_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) \end{array} \right\} \Rightarrow \begin{array}{c} C = \frac{1}{\sqrt{5}} \\ D = -\frac{1}{\sqrt{5}} \end{array}$$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \times 2^n}$$

23

Single Root Theorem

- If r is a single real root to the characteristic equation $t^2 - At - B = 0$ of a recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, $k \geq 2$, then the sequence is defined by the explicit formula:

  $a_n = Cr^n + Dnr^n$,

  where C and D are constants determined by a_0 and a_1, if given.

24