Correctness of Algorithm

CS 231
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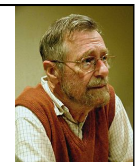
What does it mean for a program to be correct?



- Syntax errors
- Implementation errors
- Logical errors (algorithmic errors)
  - This part can be proved mathematically
  - “We now take the position that it is not only the programmer's task to produce a correct program, but also to demonstrate its correctness in a convincing manner” – Dijkstra, 1967

Predicates

- An algorithm is designed to produce a certain final state (post-condition) from a certain initial state (pre-condition).
- Proof of correctness: show that if the pre-condition is true for a collection of values, then the post-condition is also true.

Example

- Algorithm to compute a product of two nonnegative integers
  - Pre-condition: input variables x and y are non-negative integers
  - Post-condition: output variable $p = xy$

Correctness of a loop

- Method to prove the correctness of a loop
- Given a **while** loop, entry restricted by a condition G (guard).

Pre-condition for the loop
while (G)
  body
end while
Post-condition for the loop

Loop Invariant Theorem

- Given a predicate I(n), a loop is correct if:
  - Basis: I(0) is true before the first iteration of the loop
  - Inductive: For all integers $k \geq 0$, G $\wedge$ I(k) before any iteration $\rightarrow$ I(k+1) after the iteration
  - Eventual Guard Falsity: After a finite number of iterations, G becomes false
  - Correctness of post-condition: If I(N) is true when N is the least number of iterations after which G is false, the values of the algorithm variables will be as specified in the post-condition.

Loop to compute a product

Pre-condition: x and y are nonnegative integers, $i = 0$ and $product = 0$

```
while (i≠x)
    product := product + y
    i := i+1
end while
```

Post-condition: $product = xy$

Loop invariant: $I(n)$: $i = n \wedge$ product $= ny$

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Proof

- Base: $I(0)$: i=0 and product = $0*y$
- Inductive: $G \wedge I(k)$ before iteration $\rightarrow I(k+1)$ after iteration
  - inductive hypothesis:
  - $i=k \wedge product = ky$
  - inductive step:
  - $product = product + y = ky + y = (k+1)y$
  - $i = i+1 = k+1$

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Proof

- Falsity of Guard: after x iterations, $i=x$
- Correctness of Post-condition:
  - $N=x$
  - $i=N \wedge product = Ny$
  - $i=x \wedge product = xy$

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Loop Invariant

- A statement of conditions that must be true on entry into a loop and are guaranteed to remain true after every iteration of the loop
- Inductive invariant
- Finding the right one is often the hardest part of proving the correctness of a loop
- Loop invariant and negated guard implies post-condition – must be strong enough

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Loop

Pre-condition: $x = 0$, $i = 2$

```
while (i<=10)
    x := x + i*i
    i := i+1
end while
```

Post-condition: x = sum of squares of 2-10

Loop invariant: $I(n)$: $i = n \wedge x = \sum_{i=2}^{n} i^2$

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- Thinking about loops in terms of invariants help you avoid errors and bad practices:
  - off by one errors
  - wrong/missing code in the loop body
  - declarations of variables outside the loop that are only used inside the loop body

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Finding the Max Element

Pre-condition: $a_1, a_2 \ldots a_n \in Z$, $max := a_1$
for ($i :=$ 2 **to** n)
  if ($max < a_i$) **then** $max := a_i$
  next i
Post-condition:
max = the largest value in {a}

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