Strong Mathematical Induction and the Well-ordering Principle

CS 231
Dianna Xu

1

Strong induction

- Weak mathematical induction assumes P(k) is true, and uses that (and only that!) to show P(k+1) is true
- Strong mathematical induction assumes P(1), P(2), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$\left[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)\right] \rightarrow P(k+1)$$

2

Strong induction example 1

- Show that any number > 1 can be written as the product of one or more primes
- Base case: P(2)
  - 2 is the product of 2 (remember that 1 is not prime!)
- Inductive hypothesis: assume P(2), P(3), …, P(k) are all true
- Inductive step: Show that P(k+1) is true

3

Strong induction example 1

- Inductive step: Show that P(k+1) is true
- There are two cases:
  - k+1 is prime
    - It can then be written as the product of k+1
  - k+1 is composite
    - It can be written as the product of two composites, a and b, where $2 \le a \le b < k+1$
    - By the inductive hypothesis, both P(a) and P(b) are true ■

4

Strong induction vs. ordinary induction

- Determine which amounts of postage can be written with 5 and 6 cent stamps
  - Prove using both versions of induction
- Answer: any postage ≥ 20

5

Answer via mathematical induction

- Show base case: P(20):
  - 20 = 5 + 5 + 5 + 5
- Inductive hypothesis: Assume P(k) is true
- Inductive step: Show that P(k+1) is true
  - If P(k) uses a 5 cent stamp, replace that stamp with a 6 cent stamp
  - If P(k) does not use a 5 cent stamp, it must use only 6 cent stamps
    - Since k > 18, there must be four 6 cent stamps
    - Replace these with five 5 cent stamps to obtain k+1 ■

6

Answer via strong induction

- Show base cases: P(20), P(21), P(22), P(23), and P(24)
  - 20 = 5 + 5 + 5 + 5
  - 21 = 5 + 5 + 5 + 6
  - 22 = 5 + 5 + 6 + 6
  - 23 = 5 + 6 + 6 + 6
  - 24 = 6 + 6 + 6 + 6
- Inductive hypothesis: Assume P(20), P(21), …, P(*k*) are all true
- Inductive step: Show that P(*k*+1) is true
  - Obtain P(*k*+1) by adding a 5 cent stamp to P(*k*+1-5)
  - P(*k*+1-5) = P(*k*-4) is true ■

7

The Well-ordering Principle for Integers

- Let S be a set containing one or more integers all of which are greater than some fixed integer. Then S has a least element.
- Every non-empty set of positive integers contains a least element
- Equivalent to ordinary and strong mathematical inductions
  - i.e. if one is true, so are the other two

8

Archimedean property

- Let a, b be positive integers. ∃ positive integer n, such that na ≥ b.
- Assume there exists positive integers x and y such that ∀n, nx < y.
- Consider the set S = {y – nx}.
- By the well-ordering principle, S has a least element, say y-mx.
- Consider y-(m+1)x

9

Principle of mathematical induction

- Let P be a set of positive integers with the following properties:
  - 1 in P
  - k in P → k+1 in P
- Then P is the set of all positive integers

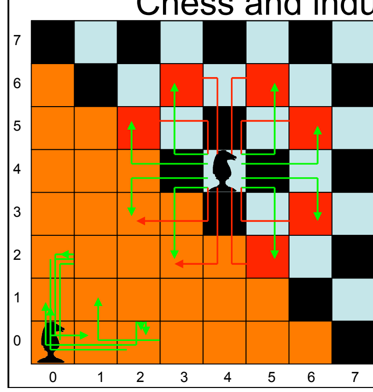
10

Proof with the well-ordering principle

- Let S be the set of all positive integers not in P.
- Assume that S is not empty.
- Then S has a least element, say a
- a > 1 (1 in P)
- a-1 is not in S (a is the least element of S)
- a-1 in P → a in P
- Contradiction ■

11

Chess and induction



Can the knight reach any square in a finite number of moves?

Show that the knight can reach any square (*i*, *j*) for which *i*+*j*=*k* where *k* > 1.

Base case: *k* = 2

Inductive hypothesis: assume the knight can reach any square (*i*, *j*) for which *i*+*j*=*k* where *k* > 1.

Inductive step: show the knight can reach any square (*i*, *j*) for which *i* +*j*=*k*+1 where *k* > 1.

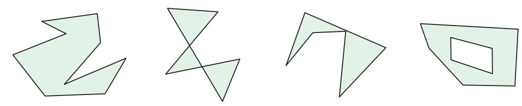
12

Chess and induction

- Inductive step: show the knight can reach any square (i, j) for which $i+j=k+1$ where $k > 1$.
  - Note that $k+1 \geq 3$, and one of i or j is ≥ 2
  - If $i \geq 2$, the knight could have moved from (i-2, j+1)
    - Since $i+j = k+1$, $i-2 + j+1 = k$, which is assumed true
  - If $j \geq 2$, the knight could have moved from (i+1, j-2)
    - Since $i+j = k+1$, $i+1 + j-2 = k$, which is assumed true ■

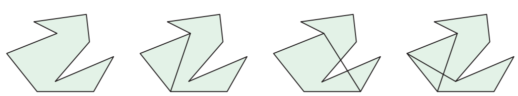
13

Polygon



14

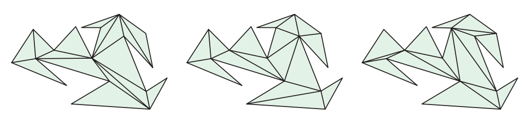
Diagonal



15

Triangulation

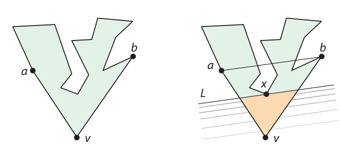
- A triangulation of a polygon is a decomposition into triangles with maximal non-crossing diagonals.



16

Existence of a Diagonal

- Every polygon with n>3 vertices has a diagonal.



17

Theorem

- Every polygon admits a triangulation.
- Every triangulation of a polygon P with n vertices has n-2 triangles and n-3 diagonals.

- Proof by strong induction

18