Mathematical Induction

CS 231

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How do you climb infinite stairs?

- Not a rhetorical question!
- First, you get to the base platform of the staircase
- Then repeat:
  - From your current position, move one step up

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What is induction?

- A method of proof: $\forall n$, $n \geq a$, $P(n)$
- Three parts:
  - Base case(s): show it is true for one element
    - $P(a)$ (get to the stair's base platform)
  - Inductive hypothesis: assume it is true for any given element
    - Assume $P(k)$, $k \geq a$ (assume you are on a stair)
  - Show that it is true for the next highest element
    - $P(k+1)$ (show you can move to the next stair)

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Why does induction work?

- Establish that the truth of a proposition follows from smaller instances of the same proposition: $P(k) \rightarrow P(k+1)$
- Establish the truth of the smallest instance: $P(a)$
- In induction, the truth percolates up through the layers to prove the whole proposition

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Induction example

- Show that the sum of the first n odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, show:

$$\forall n \ P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 = 1^2$$

$$1 = 1$$

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Induction example, continued

- Inductive hypothesis: assume true for k
  - Thus, we assume that $P(k)$ is true, or that

$$\sum_{i=1}^{k} 2i - 1 = k^2$$

  - Note: we don't yet know if this is true or not!
- Inductive step: show true for $k+1$
  - We want to show that:

$$\sum_{i=1}^{k+1} 2i - 1 = (k+1)^2$$

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Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i-1 = k^2$
- Proof of inductive step: $\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$

$$2(k+1)-1+\sum_{i=1}^{k} 2i-1$$
$$= 2(k+1)-1+k^2$$
$$= k^2+2k+1$$
$$= (k+1)^2$$

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Induction example

- Show that the sum of the first n powers of 2 is $2^{n+1}-1$, where n starts at 0.
  - Example: If $n = 4$:
  - $1+2+2^2+2^3+2^4 = 1+2+4+8+16 = 31 = 2^5-1$
  - Formally, show: $\forall n\, P(n)$ where $P(n) = \sum_{i=0}^{n} 2^i = 2^{n+1}-1$
- Base case: Show that P(0) is true

$$P(0) = \sum_{i=0}^{0} 2^0 = 1 = 2^1 - 1$$

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Induction example, continued

- Inductive hypothesis: assume true for arbitrary k
  - Thus, we assume that P(k) is true, or that

$$\sum_{i=0}^{k} 2^i = 2^{k+1}-1$$

- Inductive step: show true for k+1
  - Want to show that: $\sum_{i=0}^{k+1} 2^i = 2^{k+2}-1$

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Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=0}^{k} 2^i = 2^{k+1}-1$
- Proof of inductive step:

$$\sum_{i=0}^{k+1} 2^i = 1+2+\ldots+2^k+2^{k+1} = \sum_{i=0}^{k} 2^i + 2^{k+1}$$

$$= 2^{k+1}-1+2^{k+1} = 2\times 2^{k+1}-1 = 2^{k+2}-1 \quad \blacksquare$$

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What did we show

- Base case: P(0)
- If P(k) is true, then P(k+1) is true
  - i.e., P(k) → P(k+1)

- We know it's true for P(0)
- Because of P(k) → P(k+1), if it' s true for P(0), then it's true for P(1)
- Because of P(k) → P(k+1), if it' s true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it' s true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it' s true for P(3), then it's true for P(4)
- And onwards to infinity

- Thus, it is true for all possible values of n

$$\left[P(0) \wedge \forall k \left(P(k) \rightarrow P(k+1)\right)\right] \rightarrow \forall n\, P(n)$$

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How to do inductive proofs

- Show the base case
- Establish the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Prove the inductive step

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Another induction example

- Show that $n! < n^n, \forall n > 1$
- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$$

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Another induction example

- Show that $6 \mid n^3 - n, \forall n \in \mathbb{Z}, n \geq 0$
- Base Case: $P(0): 6 \mid 0 = 0^3 - 0$
- Inductive hypothesis: $6 \mid k^3 - k$
- Inductive step:

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= k^3 - k + 3k^2 + 3k = k^3 - k + 3(k^2 + k)$$
$$= k^3 - k + 3k(k+1) \quad \blacksquare$$

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Applications of Induction

- Algebraic (in)equalities are not the only suitable applications of induction.
- How to apply the inductive step is less obvious in non-algebraic applications of induction
  - the manipulation needed to apply the hypothesis is not algebraic

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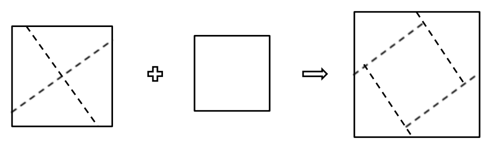
Square Cutting

- Prove that given two or more squares, one can always cut them and reform them into a large square.



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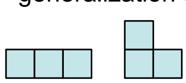
Trominoes

- Polyomino – generalization of domino
- Tromino –



- Prove that if any one square is removed from a $2^n \times 2^n$ checkerboard ($n \geq 1$, $n \in \mathbb{Z}$), the remaining squares can be completely covered by L-shaped trominoes.

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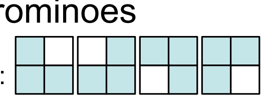
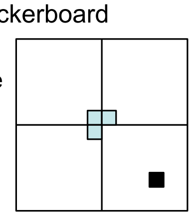
Trominoes

- P(1): 2x2 board:



- Assume a $2^k \times 2^k$ checkerboard can be covered except for any one square
- P(k+1): $2^{k+1} \times 2^{k+1}$ checkerboard
  - divide into 4 quadrants
  - each quadrant is of size $2^k \times 2^k$.



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All Horses are the Same Color

- If there is only one horse, it's of one color
- Assume within any set of *k* horses, there is only one color
- Consider *k*+1 horse, and divide into sets of {1, 2, 3, ..., *k*} and {2, 3, 4, ..., *k*, *k*+1}.
  - Each is a set of *k* horses and can be of only one color.
  - Since there is overlap among the sets, there is only one color for all *k*+1 horses.

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All Men are Bald

- A man with 0 (or 1) hair is clearly bald
- Assume a man with k hairs is bald
- One more hair on a bald head does not cure baldness, thus a man with k+1 hair is also bald.

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