1 Graphs

Definition 1 *A* **directed graph** *(or* **digraph***) G is a pair (V, E), where V is a finite set and E is a binary relation on V. The set V is called the* **vertex set** *of G, and its elements are called* **vertices** *(singular:* **vertex***). The set E is called the* **edge set** *of G, and its elements are called* **edges***.*

In an **undirected graph** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices, rather than ordered pairs.

You need to understand the following terminologies:

- endpoints
- loop
- parallel edges
- isolated vertex

Example 1 *[Vegetarians and Cannibals] There are two types of people on an island, either vegetarians or cannibals. Initially, two vegetarians and two cannibals are on the left bank of a river. With them is a boat that can hold a maximum of two people. Find a way to transport all the vegetarians and cannibals to the right bank of the river. Note that at no time can the number of cannibals on either bank outnumber the number of vegetarians.*

Solution. ⊠

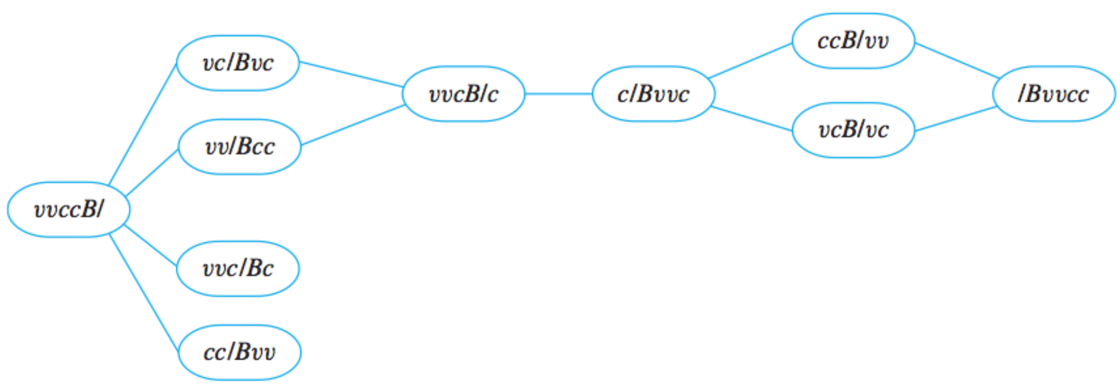

Figure 1: solution to Example 1

1.1 Special Graphs

- A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.
- Let n be a positive integer. A **complete graph** on n vertices, denoted K_n, is a simple undirected graph with n vertices and exactly one edge connecting each pair of distinct vertices.
- A **bipartite graph** is an undirected graph $G = (V, E)$ in which V can be partitioned into two sets V_1 and V_2 such that $(u, v) \in E$ implies either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$. That is, all edges go between the two sets V_1 and V_2.

- A graph $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Given a set $V' \subseteq V$, the subgraph of G **induced** by V' is the graph $G' = (V', E')$ where $E' = \{(u, v) \in E : u, v \in V'\}$.

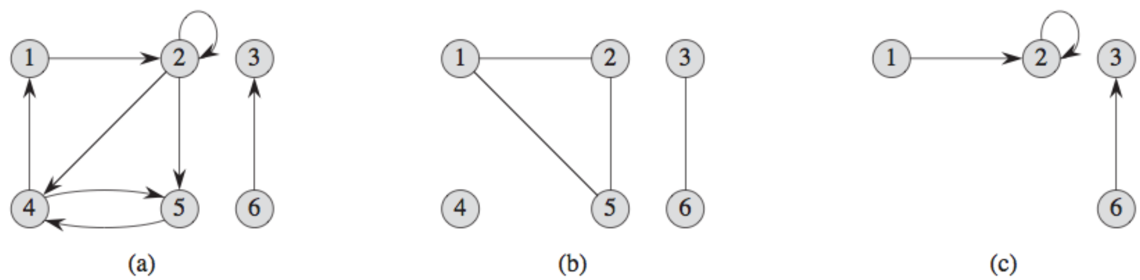

Figure 2: Directed and undirected graphs.

1.2 Degree and the Handshake Theorem

Definition 2 *The* **degree of a vertex** *in an undirected graph is the number of edges incident on it. In a directed graph, the* **out-degree** *of a vertex is the number of edges leaving it, and the* **in-degree** *of a vertex is the number of edges entering it. The* **degree** *of a vertex in a directed graph is its in-degree plus its out-degree. The* **total degree of** *G is the sum of the degrees of all the vertices of G.*

Theorem 1 (The Handshake Theorem) *If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then*

$$\text{the total degree of } G = deg(v_1) + deg(v_2) + \cdots + deg(v_n) = 2 \cdot (\text{the number of edges of } G).$$

Corollary 1 *The total degree of a graph is even.*

Example 2 *Draw a graph with the specified properties or show that no such graph exists.*

- *A graph with four vertices of degrees 1, 1, 2, and 3*
- *A graph with four vertices of degrees 1, 1, 3, and 3*
- *A simple graph with four vertices of degrees 1, 1, 3, and 3*

Theorem 2 *In any graph there are an even number of vertices of odd degree.*

Proof.[sketch]

- total degree of all vertices is even
- sum of even-degree vertices is even
- sum of odd-degree vertices is even

⊠

Is there a graph with ten vertices of degrees 1, 1, 2, 2, 2, 3, 4, 4, 4, and 6?

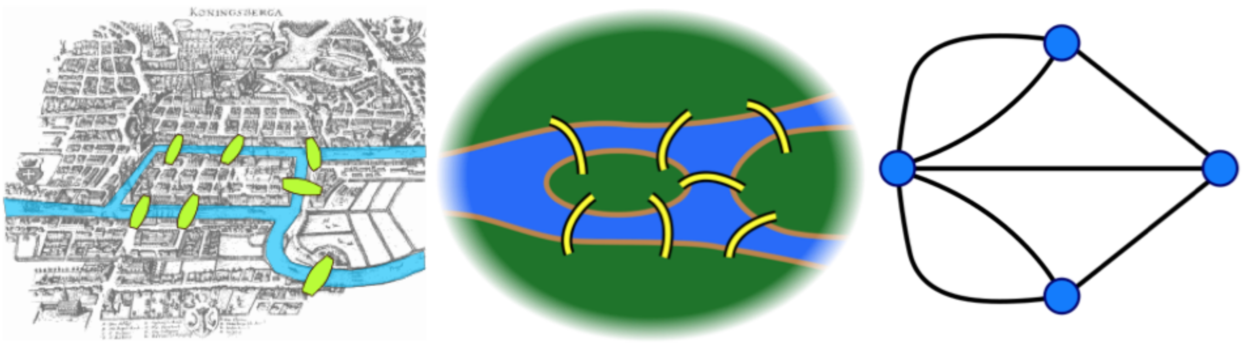

Figure 3: The Seven Bridges of Königsberg

2 Trails, Paths and Circuits

The town of Königsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks (see Figure 3). Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?

Definition 3 *Let G be a graph, and let v and w be vertices in G.*

- *A* **walk** *from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form*

$$v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \dots, n, v_{i-1}$ and v_i are the endpoints of e_i.

- *The* **trivial walk from** v **to** v *consists of the single vertex v.*
- *A* **trail from** v **to** w *is a walk from v to w that does not contain a repeated edge.*
- *A* **path from** v **to** w *is a trail that does not contain a repeated vertex.*
- *A* **closed walk** *is a walk that starts and ends at the same vertex.*
- *A* **circuit** *is a closed walk that contains at least one edge and does not contain a repeated edge.*
- *A* **simple circuit** *is a circuit that does not have any other repeated vertex except the first and last.*

2.1 Connectedness

An undirected graph is **connected** if every vertex is reachable from all other vertices. The **connected components** of a graph are the equivalence classes of vertices under the "is reachable from" relation.

Example 3 *Consider the graph in Figure 2 (b). How many connected components are there? What are they?*

An undirected graph is connected if it has exactly one connected component.

The edges of a connected component are those that are incident on only the vertices of the component; in other words, edge (u, v) is an edge of a connected component only if both u and v are vertices of the component.

A directed graph is **strongly connected** if every two vertices are reachable from each other. The **strongly connected components** of a directed graph are the equivalence classes of vertices under the "are mutually reachable" relation. A directed graph is strongly connected if it has only one strongly connected component.

Example 4 *Consider the graph in Figure 2 (a). How many strongly connected components are there?*

2.2 Euler Circuit

Definition 4 (Euler circuit) *Let G be a graph. An* **Euler circuit** *for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.*

Theorem 3 *If a graph has an Euler circuit, then every vertex of the graph has positive even degree.*

How to test whether a graph does not have an Euler Circuit?

Theorem 4 *If every vertex of a connected graph has even degree, then the graph has an Euler circuit.*

Proof. Constructive proof. See procedure Euler.

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procedure Euler(G: connected multigraph with all vertices of even degree)
  circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
  $H := G$ with the edges of this circuit removed
  while H has edges **do**
    subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of circuit
    H := H with edges of subcircuit and all isolated vertices removed
    circuit := circuit with subcircuit inserted at the appropriate vertex
  end while
  return circuit {circuit is an Euler circuit}

$\boxtimes$

From both Theorems 3 and 4, we have:

Theorem 5 *A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.*

2.3 Hamiltonian Circuits

Definition 5 *A* **Hamiltonian circuit (Hamiltonian cycle)** *of an undirected graph $G = (V, E)$ is a simple cycle that contains each vertex in V. A graph that contains a hamiltonian cycle is said to be* **hamiltonian***; otherwise, it is* **nonhamiltonian**.

What is the relationship between an Euler Circuit and a Hamiltonian Circuit?
If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

- H contains every vertex of G.
- H is connected.
- H has the same number of edges as vertices.
- Every vertex of H has degree 2.

Example 5 (A Traveling Salesman Problem) *Imagine that the drawing below is a map showing four cities and the distances in kilome- ters between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city A. Which route from city to city will minimize the total distance that must be traveled?*

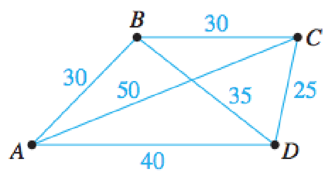


Figure 4: Traveling Salesman Problem

Solution. *Write all possible Hamiltonian circuits starting and ending at A and calculating the total distance traveled for each.*

- *ABCDA: 30+30+25+40 = 125*
- *ABDCA: 30+35+25+50 = 140*
- *ACBDA: 50+30+35+40 = 155*
- *ACDBA: 140 [A B D C A backwards]*
- *ADBCA: 155 [AC B D A backwards]*
- *ADCBA: 125 [ABCDA backwards]*

Thus either route ABCDA or ADCBA gives a minimum total distance of 125 kilometers. ⊠

The problem of finding a hamiltonian cycle in an undirected graph has been stud- ied for over a hundred years. Not all graphs are hamiltonian, however. For example, Figure 5 shows a bipartite graph with an odd number of vertices.

The **hamiltonian-cycle problem** is "Does a graph G have a hamiltonian cycle?" There is no known efficient way to determine whether a graph has a Hamiltonian circuit, or how to find one.

3 Representation of Graphs

We can choose between two standard ways to represent a graph $G = (V, E)$ as a collection of **adjacency lists** or as an **adjacency matrix**. Either way applies to both directed and undirected graphs. Because the adjacency-list representation provides a compact way to represent sparse graphs-those for which $|E|$ is much less than $|V|^2$-it is usually the method of choice.

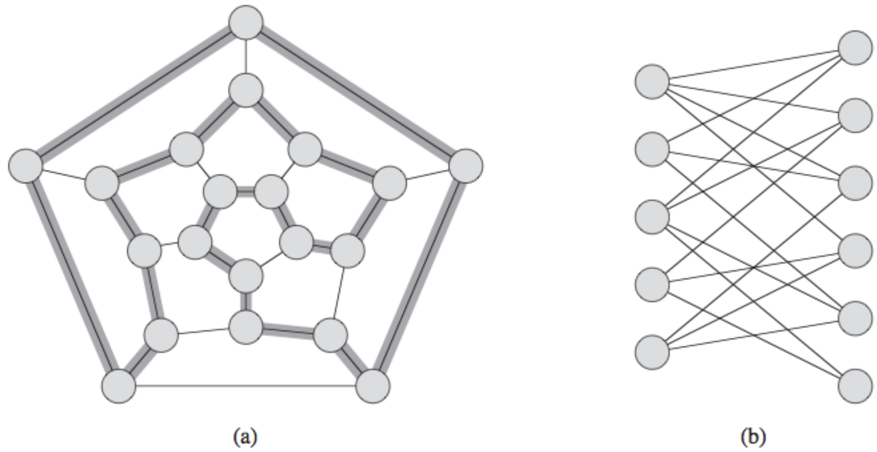


Figure 5: (a) A graph representing the vertices, edges, and faces of a dodecahedron, with a hamiltonian cycle shown by shaded edges. (b) A bipartite graph with an odd number of vertices. Any such graph is nonhamiltonian.

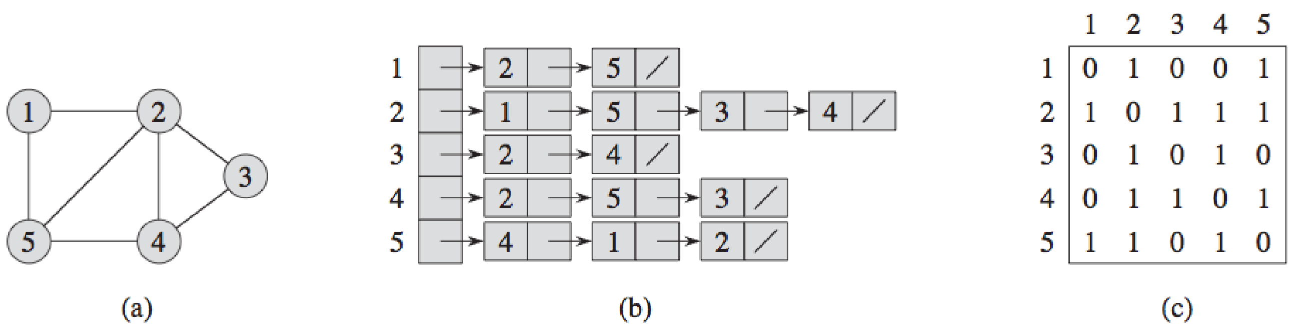


Figure 6: Two representations of an undirected graph. (a) An undirected graph G with 5 vertices and 7 edges. (b) An adjacency-list representation of G. (c) The adjacency-matrix representation of G.

4 Trees

A graph is said to be **circuit-free** if, and only if, it has no circuits. A graph is called a **tree** if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

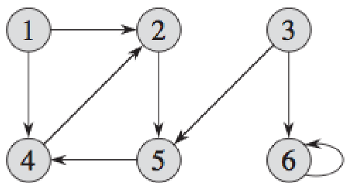
4.1 Characterizing Trees

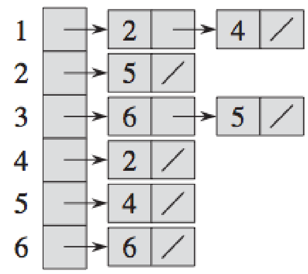
Lemma 1 *Any tree that has more than one vertex has at least one vertex of degree 1.*

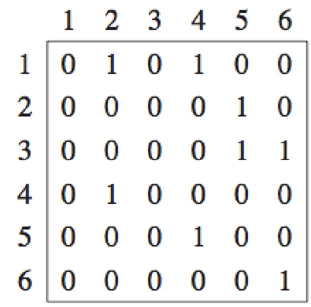
A tree vertex of degree 1 is called a **leaf**. Others are called internal vertices.

Theorem 6 *For any positive integer n, any tree with n vertices has $n-1$ edges.*

Proof.[sketch] Let $P(n)$ be "Any tree with n vertices has $n-1$ edges". Prove by induction. ⊠







	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Figure 7: Two representations of a directed graph. (a) A directed graph G with 6 vertices and 8 edges. (b) An adjacency-list representation of G. (c) The adjacency-matrix representation of G.

Lemma 2 *If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph that remains is connected.*

Theorem 7 *For any positive integer n, if G is a connected graph with n vertices and $n-1$ edges, then G is a tree.*

4.2 Rooted Trees

A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root**. The **level** of a vertex is the number of edges along the unique path between it and the root. The **height** of a rooted tree is the maximum level of any vertex of the tree.

Terminologies:

- hight
- children
- parent
- siblings
- ancestor
- descendant

A **binary tree** is a rooted tree in which every parent has *at most* two children.
Terminologies:

- left child
- right child
- full binary tree
- left subtree

- right subtree

Theorem 8 *If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of $2k+1$ vertices and has $k+1$ terminal vertices.*

Theorem 9 *For all integers $h \geq 0$, if T is any binary tree with of height h and t terminal vertices, then*

$$t \leq 2^h.$$