1 Sequences and Summation

A *sequence* is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer. For example,

$$a_m, a_{m+1}, ..., a_m$$

is a sequence where each element a_k is called a *term*. In the above sequence, a_m is the initial term and a_n is the final term. We may have infinite sequences. For example,

$$a_m, a_{m+1}, ...$$

is such an infinite sequence.

Example 1 *Show that* $(1+2+3+\cdots+n)+(1+2+3+\cdots+n+(n+1))=(n+1)^2$.

Example 2 *Show that* $(1+2+3+\cdots+(n-1))^2+n^3=(1+2+3+\cdots+n)^2$.

Example 3 *Show that* $1^3+2^3+\cdots+n^3=(1+2+3+\cdots+n)^2$.

Example 4 *Show that* $1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Zeno's paradox of motion: "Zeno explains that to reach the wastebasket, the ball must first travel half the distance from my hand to the basket, and then from there, half the remaining distance, and so on. Hance the ball must pass through an infinite number of locations and travel an infinite number of distance. How can the ball do an infinite number of things in a finite amount of time? Motion, Zeno concludes, must be an illusion." – from Rediscovering Mathematics.

This leads to an infinite sum $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\ldots$. What is its value?

Exercise 1 *Using the identity*

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

find the value of the infinite sum $\frac{1}{1*2}+\frac{1}{2*3}+\frac{1}{3*4}+\ldots$.

Definition 1 (Factorial) *For each positive integer n, the quantity n* factorial *denoted $n!$, is defined to be the product of all the integers from 1 to n:*

$$n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1.$$

Zero factorial*, denoted* $0!$*, is defined to be 1:*

$$0! = 1.$$

Definition 2 (n choose r) *Let n and r be integers with $0\le r\le n$. The symbol*

$$\binom{n}{r}$$

is read "n choose r" and represents the number of subsets of size r that can be chosen from a set with n elements.

For all integers n and r with $0\le r\le n$,

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

2 Number Systems

[Chapter 2.5, 5.1]

2.1 Decimal Representation

Power of 10	10^5	10^4	10^3	10^2	10^1	10^0
Decimal Form	100000	10000	1000	100	10	1

2.2 Binary Representation

Power of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Binary Form	1024	512	256	128	64	32	16	8	4	2	1

Example 5 *Represent* 110101_2 *in decimal notation.*

$$\begin{aligned}
110101_2 &= 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\
&= 32 + 16 + 4 + 1 \\
&= 53_{10}
\end{aligned}$$

Example 6 *Represent* 38_{10} *in binary notation.*

1. *Using Binary Representation table above:*

$$\begin{aligned}
38_{10} &= 32 + 4 + 2 \\
&= 0 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
&= 00100110_2
\end{aligned}$$

2. *Using repeated division by 2:*

$$\begin{aligned}
38 &= 19 \cdot 2 + \mathbf{0}, \\
19 &= 9 \cdot 2 + \mathbf{1}, \\
9 &= 4 \cdot 2 + \mathbf{1}, \\
4 &= 2 \cdot 2 + \mathbf{0}, \\
2 &= 1 \cdot 2 + \mathbf{0}, \\
1 &= 0 \cdot 2 + \mathbf{1}.
\end{aligned}$$

By repeated substitution, then,

$$\begin{aligned}
38 &= 19 \cdot 2 + \mathbf{0} \\
&= (9 \cdot 2 + \mathbf{1}) \cdot 2 + \mathbf{0} = 9 \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} \\
&= (4 \cdot 2 + \mathbf{1}) \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} = 4 \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} \\
&= (2 \cdot 2 + \mathbf{0}) \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} = 2 \cdot 2^4 + \mathbf{0} \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} \\
&= (1 \cdot 2 + \mathbf{0}) \cdot 2^4 + \mathbf{0} \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0} \\
&= \mathbf{1} \cdot 2^5 + \mathbf{0} \cdot 2^4 + \mathbf{0} \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0}
\end{aligned}$$

Therefore, $38_{10} = 100110_2$.

2.3 Two's Complements and the Computer Representation of Negative Integers

Definition 3 *Given a positive integer a, the two's complement of a relative to a fixed bit length n is the n-bit binary representation of*

$$2^n - a.$$

Note that $2^8 - a = [(2^8 - 1) - a] + 1$.
To find the 8-bit two's complement of a positive integer a that is at most 255:

- Write the 8-bit binary representation for a.
- Flip the bits.
- Add 1 in binary notation.

Example 7 *Find the 8-bit two's complement of 19.*

$$19_{10} = (16 + 2 + 1)_{10} = 00010011_2 \xrightarrow{\textit{flip the bits}} 11101100 \xrightarrow{\textit{add 1}} 11101101$$

Example 8 *Use 8-bit representations to compute* $39 + (-89)$.
There are 3 steps:

1. *Using the 8-bit representation to represent* 39 *and* -89.
   - $39_{10} = (32 + 4 + 2 + 1)_{10} = 100111_2$. *So the 8-bit representation of 39 is 00100111.*
   - *Since the 8-bit representation of* -89 *is the two's complement of 89, we have*

   $$89_{10} = (64 + 16 + 8 + 1)_{10} = 01011001_2 \xrightarrow{\textit{flip the bits}} 10100110 \xrightarrow{\textit{add 1}} 10100111.$$

   So the 8-bit representation of -89 *is 10100111.*

2. *Add the 8-bit representations in binary notation and truncate the 1 in the* 8^{th} *position if there is one:*

$$\begin{array}{rcccccccc} & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ + & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

3. *Find the decimal equivalent of the result. Since its leading bit is 1, this number is the 8-bit representation of a negative integer.*

$$11001110 \xrightarrow{\textit{flip the bits}} 00110001 \xrightarrow{\textit{add 1}} 00110010 \leftrightarrow -(32 + 16 + 2)_{10} = -50_{10}$$

2.4 Hexadecimal Notation

Decimal	Hexadecimal	4-Bit Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Example 9 *Convert* $B09F_{16}$ *to binary notation.*

$$\begin{array}{cccc} B & 0 & 9 & F \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1011 & 0000 & 1001 & 1111 \end{array}$$

3 Method of Proof by Mathematical Induction

Proofs by induction is the most common method of proof in Computer Science. To prove a property $P(n)$ defined for integers n, if the following two statements are true:

1. $P(a)$ is true for a fixed integer a.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

then the statement

$$\text{for all integers } n \geq a, P(n)$$

is true.

Example 10 *How many pieces do you get from cutting a circle with n distinct cuts? That is, every cut has to go through every other cut exactly at one place and not at the same spot. A chart describes the number of cuts and the pieces after the cuts is below.*

cuts	*pieces*
1	2
2	4
3	7
4	11
...	...
n	?

#	T_n
1	1
2	3
3	6
4	10
...	...
n	T_n

T_n is the triangle number. So there are $T_n + 1$ pieces for n cuts.

Theorem 1 *$P_n = T_n + 1$.*

We will prove it by mathematical induction. Note that whenever a line is added, if there were n lines, then there will be $n+1$ new pieces. Let P_n be the number of pieces with n cuts. Then $P_n = P_{n-1} + n$. This is a recurrence relation. We assume that the theorem is true for a case and we have to come up with an argument to show that the next case is true.

Proof. *Base case: $T_1 = 1$ and $P_1 = 2 = T_1 + 1$. Assume that $P_n = T_n + 1$. We need to prove that $P_{n+1} = T_{n+1} + 1$.*

$$\begin{aligned} P_{n+1} &= P_n + n + 1 \\ &= (T_n + 1) + n + 1 \\ &= (T_n + n + 1) + 1 \\ &= T_{n+1} + 1 \end{aligned}$$

$\boxtimes$

To prove a goal of the form $\forall n \in \mathbb{N} P(n)$:

First prove $P(0)$, and then prove $\forall n \in \mathbb{N}(P(n) \to P(n+1))$. The first of these proofs is sometimes called the *base case* and the second the *induction step*.

Form of the Proof
Base case: [Proof of $P(0)$ goes here.]
Induction step: [Proof of $\forall n \geq a(P(n) \to P(n+1))$]

Definition 4 (Closed Form) *If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written in* closed form.

Example 11 *Consider the sequence $2^0 + 2^1 + \cdots + 2^n$, is there any pattern w.r.t the value of the sum? What is the closed form for this sequence?*

$$\begin{aligned} 2^0 &= 1 = \\ 2^0 + 2^1 &= 1 + 2 = \\ 2^0 + 2^1 + 2^2 &= 1 + 2 + 4 = \\ 2^0 + 2^1 + 2^2 + 2^3 &= 1 + 2 + 4 + 8 = \end{aligned}$$

Theorem 2 *For every natural number n, $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.*

Example 12 *Once you understand why mathematical induction works, you should be able to understand proofs that involve small variations on the method of induction. This example illustrates such a variation.*

In this example we'll try to figure out which is larger, n^2 or 2^n. Let's try out a few values of n:

n	n^2	2^n	*larger one*
0	*0*	*1*	2^n
1	*1*	*2*	2^n
2	*4*	*4*	*tie*
3	*9*	*8*	n^2
4	*16*	*16*	*tie*
5	*25*	*32*	2^n
6	*36*	*64*	2^n

Theorem 3 *For every natural number* $n \geq 5$ *,* $2^n > n^2$*.*

Exercise 2 *Prove that for all* $n \geq 10$*,* $2^n > n^3$*.*

Exercise 3 *Guess the number of different ways for n people to arrange themselves in a straight line, and prove your guess is correct by induction.*

Exercise 4 *Guess a formula for the sum below, and prove you are right by induction.*

$$1 + 1(2) + 2(3) + 3(4) + \cdots + n(n+1)$$

Exercise 5 *Find a formula for* $3^0 + 3^1 + 3^2 + \cdots + 3^n$*, for* $n \geq 0$*, and prove that your formula is correct.*

Theorem 4 (Covering Board with Trominoes) *For any positive integer n, a* $2^n \times 2^n$ *square grid with any one square removed can be covered with L-shaped tiles.*

4 Strong Induction

In the induction step of a proof by mathematical induction, we prove that a natural number has some property based on the assumption that the previous number has the same property. In some cases this assumption isn't strong enough to make the proof work, and we need to assume that all smaller natural numbers have the property. This is the idea behind a variant of mathematical induction sometimes called strong induction:

Principle of Strong Mathematical Induction
Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a)$, $P(a+1)$, $\ldots$, and $P(b)$ are all true. (basis step)
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. (inductive step)

Then the statement

$$\text{for all integers } n \geq a, P(n)$$

is true. The **induction hypothesis** is the supposition that $P(i)$ is true for all integers i from a through k is called the inductive hypothesis. That is, $P(a), P(a+1), \ldots, P(k)$ are all true.
To prove a goal of the form $\forall n \in \mathbb{N} P(n)$**:**
Prove that $\forall n[(\forall k < n P(k)) \rightarrow P(n)]$, where both n and k are natural numbers.

Form of the Proof
Let n be an arbitrary natural number, assume that $\forall k < nP(k)$, and then prove $P(n)$.

Example 13 *Prove that every integer $n > 1$ is either prime or a product of primes.*
Proof sketch:
Goal: $\forall n \in \mathbb{N}[n > 1 \rightarrow (n$ *is prime* $\vee n$ *is a product of primes*$)]$.
Induction hypothesis: $\forall k < n[k > 1 \rightarrow (k$ *is prime* $\vee k$ *is a product of primes*$)]$.
We must show that $n > 1 \rightarrow (n$ *is prime* $\vee n$ *is a product of primes*$)]$.
Assume $n > 1$, and according to our strategies for proving disjunctions, a good way to complete the proof would be to assume that n is not prime and prove that it must be a product of primes. Because the assumption that n is not prime means $\exists a \exists b(n = ab \wedge a < n \wedge b < n)$, we immediately use existential instantiation to introduce the new variables a and b into the proof. Applying the inductive hypothesis to a and b now leads to the desired conclusion.

Example 14 *Show that if F_n is the n^{th} Fibonacci number, then*

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Example 15 (Well-ordering principle) *Every nonempty set of natural numbers has a smallest element.*
Proof sketch:
Goal: $\forall S \subseteq \mathbb{N}[S \neq \varnothing \rightarrow S$ *has a smallest element*$)$.
After letting S be an arbitrary subset of $\mathbb{N}$, we'll prove the contrapositive of the conditional statement. In other words, we will assume that S has no smallest element and prove that $S = \varnothing$. The way induction comes into it is that, for a set $S \subseteq \mathbb{N}$, to say that $S = \varnothing$ is the same as saying that $\forall S \subseteq \mathbb{N}(n \notin S)$. We'll prove this last statement by strong induction.

In the next example, we show how the well-ordering principle can be used in the proof.

Example 16 $\sqrt{2}$ *is irrational.*
Proof. *Suppose that $\sqrt{2}$ is rational. This means that $\exists q \in \mathbb{Z}^+ \exists p \in \mathbb{Z}^+(p/q = \sqrt{2})$, so the set $S = \{q \in \mathbb{Z}^+ \mid \exists p \in \mathbb{Z}^+(p/q = \sqrt{2})\}$ is nonempty. By the well-ordering principle we let q be the smallest element of S. Since $q \in S$, we can choose some $p \in \mathbb{Z}^+$ such that $p/q = \sqrt{2}$. Therefore $p^2/q^2 = 2$, so $p^2 = 2q^2$ and therefore p^2 is even. We now apply the theorem which says that for any integer x , x is even iff x^2 is even. Since p^2 is even, p must be even, so we can choose some $p' \in \mathbb{Z}^+$ such that $p = 2p'$. Therefore $p^2 = 4p^2$, and substituting this into the equation $p^2 = 2q^2$ we get $4p'^2 = 2q^2$, so $2p'^2 = q^2$ and therefore q^2 is even. Again, this means q must be even, so we can choose some $q' \in \mathbb{Z}^+$ such that $q = 2q'$. But then $2 = p/q = (2p')/(2q') = p'/q'$, so $q' \in S$. Clearly $q' < q$, so this contradicts the fact that q was chosen to be the smallest element of S. Therefore, $\sqrt{2}$ is irrational.* ⊠

5 Recursion

5.1 Recurrence Equation

Compound Interest: Start with X dollars at 10% year. The number of dollars after the n^{th} year equals 1.1 times the number of dollars after the previous year. That is, $D(n) = 1.1D(n-1)$, and $D(0) = X$.

This is called a *recurrence equation*. Formally,

Definition 5 *A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k1}, a_{k2}, \ldots, a_{ki}$, where i is an integer with $ki \geq 0$. The initial conditions for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i1}$, if i is a fixed integer, or $a_0, a_1, a_2, \ldots, a_m$, where $m \geq 0$ is an integer, if i depends on k.*

The easiest and most common sense way to solve a recurrence equation is to use iteration (repeated substitution):

$$\begin{aligned}
D(n) &= 1.1D(n-1) \text{ use } D(n-1) = 1.1D(n-2) \text{ to substitute for } D(n-1) \text{, we have} \\
D(n) &= 1.1^2 D(n-1) \text{ and so on...} \\
&\ldots \\
D(n) &= 1.1^r D(n-r)
\end{aligned}$$

Let $r = n$, we have $D(n) = 1.1^n D(0) = 1.1^n X$.

Recursion, mathematical induction, and recurrence equations are closely related in the following way: The algorithm uses recursion, the proof of the correctness of the algorithm uses mathematical induction, and the analysis of the time requirements results in a recurrence equation.

Example 17 (Binary Search) *Given a sorted array of n numbers and a specific number k, find out the index of k. The running time of this algorithm, denoted by $T(n)$, is:*

5.2 The Tower of Hanoi

On the steps of the altar in the temple of Benares, for many, many years Brahmins have been moving a tower of 64 golden disks from one pole to another; one by one, never placing a larger on top of a smaller. When all the disks have been transferred the Tower and the Brahmins will fall, and it will be the end of the world.

How do we solve this problem? Our plan is as follows:

- Review the solution of this problem, write the pseudocode, then play a couple of examples so that every one understands the problem.
- Analyze the time complexity - how long the algorithm takes to run. A geometric series will show up and we will see how to solve it in this version.
- Show a graph representation of the computation.

1. Recursive solution (Pseudocode):

Algorithm 1 Tower of Hanoi (Input: n disks, three posts $From, To, Using$)

ToH($n, From, To, Using$)

2. Analyze the time complexity:

$$\begin{aligned} T(0) &= 0 \\ T(n) &= 2T(n-1)+1 \\ &= 2(2T(n-2)+1)+1 = 2^2T(n-2)+2+1 \\ &= 2^2(2T(n-3)+1)+2+1 = 2^3T(n-3)+2^2+2+1 \\ &= \ldots \\ &= 2^rT(n-r)+2^{r-1}+2^{r-2}+\cdots+2+1 \end{aligned}$$

Let $r = n$, then $T(n) = 2^nT(0)+2^{n-1}+2^{n-2}+\cdots+2+1 = 2^{n-1}+2^{n-2}+\cdots+2+1$. This is a *geometric series* since each successive terms is multiplied by a specific number.
$2T(n) = 2^n+2^{n-1}+2^{n-2}+\cdots+2$. Therefore, $T(n) = 2^n-1$.

3. A graph representation.

   - Each configuration is represented as a vertex.
   - If you can move one disk from one configuration to another configuration, then there is an edge between these two configurations.

When we have one disk with 3 pegs. There are 3 configurations. This is shown in Figure 1.

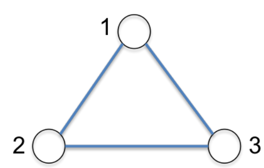

Figure 1: Tower of Hanoi - 1 disk with 3 pegs

When there are 2 disks with 3 pegs. Each configuration has 2 disks.

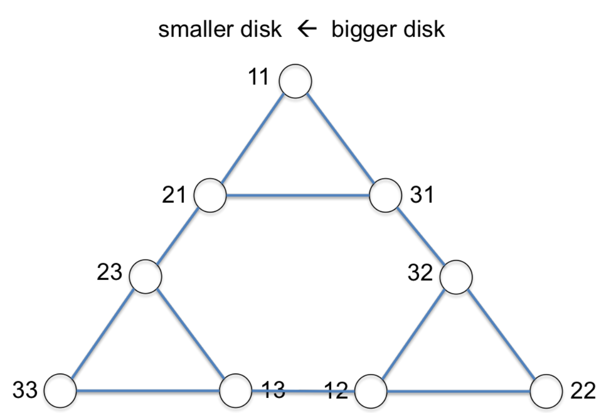

Figure 2: Tower of Hanoi - 2 disks with 3 pegs

What does it happen when we have 3 disks with 3 pegs?

The next two examples further show how to use repeated substitution to solve recurrence equations.

Example 18 *Solve the recurrence equation* $T(n) = 2T(\frac{n}{2}) + n$ *where* $T(1) = 1$.

Example 19 *Solve the recurrence equation* $T(n) = 3T(\frac{n}{2}) + n$ *where* $T(1) = 1$.

5.3 Recursion and Structure Induction

Consider the following recursive definition:

- $f(0) = 0$
- $f(n) = 1 + f(n-2)$

Is it a good definition?

In this section we discuss recursive definitions for sets and functions. We also introduce *structural induction*, a version of mathematical induction that is used to prove properties of recursively defined sets.

Recursive definition for a set

1. BASE: A statement that certain objects belong to the set.
2. RECURSION: A collection of rules indicating how to form new set objects from those already known to be in the set.
3. RESTRICTION: A statement that no objects belong to the set other than those coming from 1 and 2.

Example 20 *Use recursive definitions to define the following sets:*

- *The set of positive integers.*
- *The set of odd positive integers.*
- *The set of positive integer powers of 3*

Recursive definition of Boolean expressions:

1. BASE: Each symbol of the alphabet is a Boolean expression.
2. RECURSION: If P and Q are Boolean expressions, then so are $(P \wedge Q)$ and $(P \vee Q)$ and $\neg P$.
3. RESTRICTION: There are no Boolean expressions over the alphabet other than those obtained from 1 and 2.

To prove that every object in a recursively defined set satisfies a given property, we use **structural introduction** as follows:

Let S be a set that has been defined recursively, and consider a property that objects in S may or may not satisfy. To prove that every object in S satisfies the property:

1. Show that each object in the BASE for S satisfies the property;

2. Show that for each rule in the RECURSION, if the rule is applied to objects in S that satisfy the property, then the objects defined by the rule also satisfy the property.

Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, it must be the case that every object in S satisfies the property.

Example 21 *Show that $F(n) < 2^n$ where $F(n)$ is the n^{th} Fibonacci number using mathematical induction, strong induction and structural induction. The recursive definition of Fibonacci number is*

- *Base: $F(1) = 1$ and $F(2) = 1$*
- *Recursion: $F(n) = F(n-1) + F(n-2)$*

Example 22 *Parenthesized Expressions*

1. *A sequence of n+1 matrices $A_1 A_2 \dots A_{n+1}$ can be multiplied together in many different ways dependent on the way n pairs of parentheses are inserted. For example for n+1 = 3, there are two ways to insert the parentheses: $((A_1 A_2) A_3)$ and $(A_1 (A_2 A_3))$. Write a recurrence equation for the number of ways to insert k pairs of parenthesis. Do not solve it. (Hint: Concentrate on where the last multiplication occurs).*

2. *Write a list of the different ways to parenthesize a sequence of n+1 matrices for n+1=2,3,4.*

3. *A balanced arrangement of parenthesis is defined inductively as follows:*

   The empty string is a balanced arrangement of parentheses. If x is balanced arrangement of parentheses then so is (x). If u and v are each a balanced arrangement of parentheses, then so is uv.

   Write a list of strings that represent a balanced arrangement of n parentheses for n=1,2,3.

4. *Describe a 1-1 correspondence between the strings that are balanced arrangements of n pairs of parentheses, and the number of ways to multiply a sequence of n+1 matrices.*

6 Triangulation

An art gallery has several rooms. For security reasons, each room is guarded by cameras that see in all directions. How can we have as few cameras as possible to cover the whole gallery (see Figure 3)?

A **polygon** is a closed geometric figure consisting of a sequence of line segments $s_1, s_2, \dots, s_n$, called **sides**.Each pair of consecutive sides,s_i and s_{i+1}, for $i = 1, 2, \dots, n-1$, as well as the last side s_n and the first side s_1, of the polygon meet at a common endpoint, called a **vertex**.

- A polygon is called **simple** if no two nonconsecutive sides intersect.
- Every simple polygon divides the plane into two regions: its **interior**, consisting of the points inside the curve, and its **exterior**, consisting of the points outside the curve.
- A polygon is called **convex** if every line segment connecting two points in the interior of the polygon lies entirely inside the polygon.
- A polygon that is not convex is said to be **nonconvex**. In Figure 4, polygons (a) and (b) are convex, but polygons (c) and (d) are not.

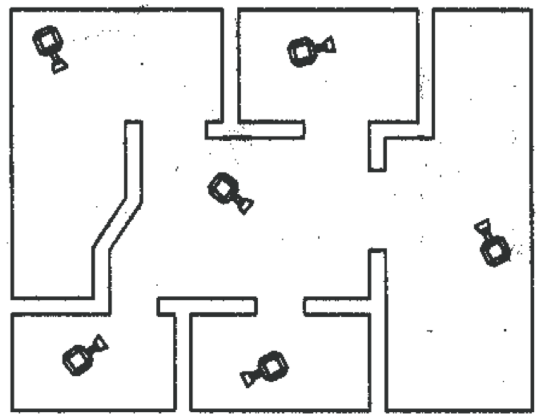

Figure 3: An Art Galley with Cameras

- A **diagonal** of a simple polygon is a line segment connecting two nonconsecutive vertices of the polygon, and a diagonal is called an **interior diagonal** if it lies entirely inside the polygon, except for its endpoints.

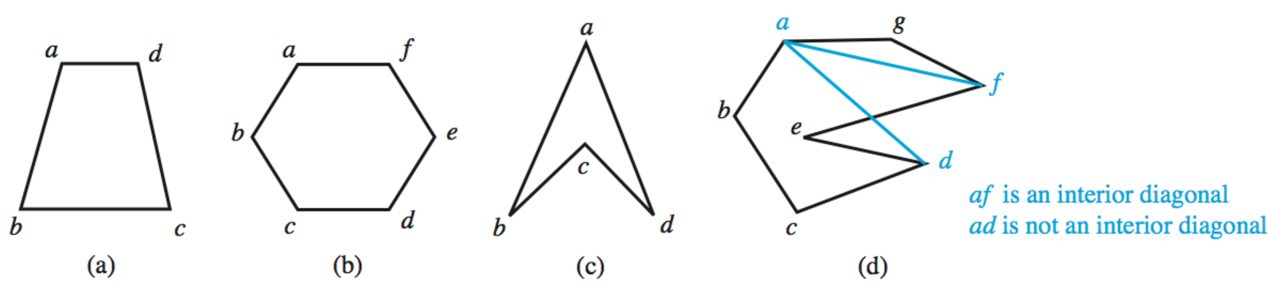

Figure 4: Convex and Nonconvex Polygons.

A **triangulation of a simple polygon** P is the decomposition of P into triangles by a maximal set of non-intersecting diagonals. Figure 5 shows a polygon and one triangulataion of this polygon.

Lemma 1 *Every simple polygon with at least four sides has an interior diagonal.*

Theorem 5 *A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n-2$ triangles.*

Proof. We will prove this result using strong induction. Let $T(n)$ be the statement that every simple polygon with n sides can be triangulated into $n-2$ triangles.

Base case: $T(3)$ is true because a simple polygon with three sides is a triangle. We do not need to add any diagonals to triangulate a triangle; it is already triangulated into one triangle, itself. Consequently, every simple polygon with $n = 3$ has can be triangulated into $n-2 = 3-2 = 1$ triangle.

Induction step: Assume that $T(j)$ is true for all integers j with $3 \leq j \leq k$. That is, we assume that we can triangulate a simple polygon with j sides into $j-2$ triangles whenever $3 \leq j \leq k$. Now suppose that we have a simple polygon P with $k+1$ sides. Because $k+1 \geq 4$, by Lemma 1, P has an interior

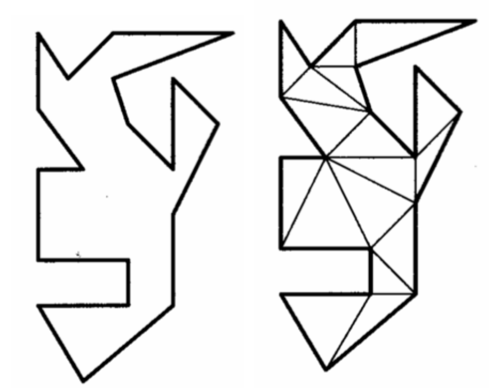

Figure 5: A Polygon and a Triangulation.

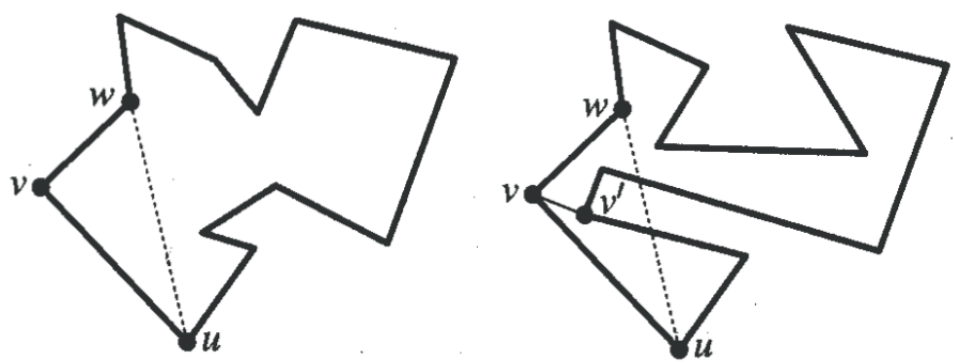

Figure 6: Existence of a Diagonal.

diagonal ab. Now, ab splits P into two simple polygons Q, with s sides, and R, with t sides. The sides of Q and R are the sides of P, together with the side ab, which is a side of both Q and R. Note that $3 \leq s \leq k$ and $3 \leq t \leq k$ because both Q and R have at least one fewer side than P does (after all, each of these is formed from P by deleting at least two sides and replacing these sides by the diagonal ab). Furthermore, the number of sides of P is two less than the sum of the numbers of sides of Q and the number of sides of R, because each side of P is a side of either Q or of R, but not both, and the diagonal ab is a side of both Q and R, but not P. That is, $k+1 = s+t-2$.

We now use the inductive hypothesis. Because both $3 \leq s \leq k$ and $3 \leq t \leq k$, by the inductive hypothesis we can triangulate Q and R into $s-2$ and $t-2$ triangles, respectively. Next, note that these triangulations together produce a triangulation of P . (Each diagonal added to triangulate one of these smaller polygons is also a diagonal of P .) Consequently, we can triangulate P in to a total of $(s-2)+(t-2) = s+t-4 = (k+1)-2$ triangles.This completes the proof by strong induction. That is, we have shown that every simple polygon with n sides, where $n \geq 3$, can be triangulated into $n-2$ triangles. ⊠

7 Josephus Problem

There are people standing in a circle waiting to be executed. The counting out begins at some point in the circle and proceeds around the circle in a fixed direction. In each step, a certain number of people are skipped and the next person is executed. The elimination proceeds around the circle (which is becoming smaller and smaller as the executed people are removed), until only the last person remains, who is given freedom.

The task is to choose the place in the initial circle so that you are the last one remaining and so survive. In the following, n denotes the number of people in the initial circle and the people in the circle are numbered from 1 to n.

1. Experiment with smaller numbers and come up the recursive equations:

   Consider the problem in terms of a smaller problem and see how you can find the solution for the current problem based on the smaller problem.

   Observation: *Consider a number of people that is the double of another number. After finishing a cycle, half of the people are gone and the problem now is similar as the original problem, except that people are renumbered.*

   Draw a graph and see what is the solutions to the problem when there are 3 people and 6 people, or 7 people and 14 people. Find out the relationship between the results.

   - $J(3) = 1$ and $J(6) = 5$
   - $J(7) = 7$ and $J(14) = 13$

   Recursive equations:

   - $J(2n) = 2J(n) - 1$
   - $J(2n + 1) = 2J(n) + 1$

2. From base cases to a general formula:
   We would like to have a formula that solves the problem with n people given n as the parameter.

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15

Note that the solutions repeat. When do they start to repeat? How do they repeat? What would be the formula that specifies the relation between n and $J(n)$?

Theorem 6 *$J(2^m + p) = 2p + 1$ where 2^m is the biggest power of 2 that n has and $p < 2^m$.*

3. Binary representation of the problem:
   For the same problem, let's write out the binary values of n and $J(n)$. Do you see any patterns here? Can you derive $J(n)$ from n?

n	$J(n)$
1	1
10	1
11	11
100	1
101	11
110	101
111	111
1000	1
1001	11
1010	101
1011	111
1100	1001
1101	1011
1110	1101
1111	1111

Exercise 6 *Write a recurrence relation to compute the number of binary strings with n digits that do not have two consecutive 1's. Solve the recurrence, and determine what percentage of 8-bit binary strings do not contain two consecutive 1's.*