1 Predicates and Quantifiers

We have seen how to represent properties of objects. For example, $B(x)$ may represent that x is a student at Bryn Mawr College. Here B stands for "is a student at Bryn Mawr College" and x is a free variable that may be true for some values of x and false for others. As another example, if we want to express "x is a friend of y", we could use $F(x, y)$ where F stand for "is a friend of" and both x and y are free variables. In these two examples, B and F are called *predicate symbols* and x and y are called *predicate variables.* A *predicate* is a predicate symbol together with suitable predicate variables.

Definition 1 *A* predicate *is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The* domain *of a predicate variable is the set of all values that may be substituted in place of the variable.*

Sometimes we would like to express whether a predicate $P(x)$ is true for all values of x or some values of x. To express these ideas, we introduce two symbols, called *quantifiers*.

- Universal quantifier $\forall$: To say that the predicate $P(x)$ is true for all possible valuse of x, we write $\forall x P(x)$, which is read "For all x, $P(x)$".
- Existential quantifier $\exists$: To say that there exists or there is at least one value of x in the universe for which the predicate $P(x)$ is true, we use $\exists x P(x)$ and read "there exists an x such that $P(x)$". The statement $\exists x P(x)$ is true means that the truth set is not equal to $\varnothing$.

For example, if we want to express "All human beings are mortal", using the predicate $M(x)$ to denote "human being x is mortal and H being the set of all human beings, then we write $\forall x \in H, M(x)$. The statement $\forall x \in U, P(x)$ is true when the truth set of $P(x)$ is the whole universe U.

Example 1 *What do the following formulas mean? Are they true or false?*

1. $\forall x(x^2 \geq 0)$, *where the universe of discourse is* $\mathbb{R}$, *the set of all real numbers.*
2. $\exists x(M(x) \wedge B(x))$, *where the universe of discourse is the set of all people,* $M(x)$ *stands for the statement "x is a man," and* $B(x)$ *means "x has brown hair."*
3. $\forall x(M(x) \to B(x))$, *with the same universe and the same meanings for* $M(x)$ *and* $B(x)$.
4. $\forall x L(x, y)$, *where the universe is the set of all people, and* $L(x, y)$ *means "x likes y."*

Note that in the statement $\forall x L(x, y)$, variable x is a bound variable and y is free. In general, even if x is a free variable in some statement $P(x)$, it is a bound variable in the statements $\forall x P(x)$ and $\exists x P(x)$. Because of this, we say that quantifiers *bind* variables. Recall that a bound variable can always be replaced with a new variable without changing the meaning of the statement. Therefore $\forall x L(x, y)$ is equivalent to $\forall z L(z, y)$. Both mean that everyone likes y. Words *all*, *every*, *every one*, and *everything* are usually an indication of using the universal quantifier $\forall$, and words *someone*, *exist*, and *something* usually indicate that the existantial quantifier $\exists$ needs to be used.

Example 2 *Analyze the logical forms of the following statements.*

1. *Someone didn't do the homework.*
2. *Everything in that store is either overpriced or poorly made.*

3. *Susan likes everyone who dislikes Joe.*
4. $A \subseteq B$.

Example 3 *Analyze the logical forms of the following statements.*

1. *Some students are married.*
2. *All parents are married.*
3. *Nobody likes a sore loser.*
4. *If a person in the dorm has a friend who has the measles, then everyone in the dorm will have to be quarantined.*

Exercise 1 *Analyze the logical forms of the following statements.*

1. *Nobody's perfect.*
2. $A \cap B \subseteq B \setminus C$.
3. *If $A \subseteq B$, then A and $C \setminus B$ are disjoint. Two sets are said to be disjoint if, and only if, they have no elements in common. Symbolically, S_1 and S_2 are disjoint iff $S_1 \cap S_2 = \varnothing$.*

Example 4 *What do the following statements mean? Are they true or false? The universe of discourse in each case is $\mathbb{R}$, the set of all real numbers.*

1. $\forall x \exists y(x + y = 3)$
2. $\exists y \forall x(x + y = 3)$

Example 5 *What do the following statements mean? Are they true or false? The universe of discourse in each case is $\mathbb{N}$, the set of all natural numbers.*

1. $\forall x \exists y(x < y)$
2. $\exists y \forall x(x < y)$
3. $\exists x \forall y(x < y)$
4. $\forall y \exists x(x < y)$
5. $\exists x \exists y(x < y)$
6. $\forall x \forall y(x < y)$

2 Equivalence Involving Quantifiers

Quantifier Negation laws

$\neg\exists x P(x)$	is equivalent to	$\forall x \neg P(x)$
$\neg\forall x P(x)$	is equivalent to	$\exists x \neg P(x)$

Example 6 *Negate these statements and then reexpress the results as equivalent positive statements.*

1. $A \subseteq B$.

2. *Every student has a course he took and he doesn't like.*

Necessary and Sufficient Conditions, Only If		
$\forall x, r(x)$ is a **sufficient condition** for $s(x)$	means	"$\forall x$, if $r(x)$ then $s(x)$"
$\forall x, r(x)$ is a **necessary condition** for $s(x)$	means	"$\forall x$, if not $r(x)$ then not $s(x)$", or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$"
$\forall x, r(x)$ **only if** $s(x)$	means	"$\forall x$, if not $s(x)$ then not $r(x)$", or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$"

Example 7 *Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.*

1. *Squareness is a sufficient condition for rectangularity.*

2. *Being at least 35 years old is a necessary condition for being President of the United States.*

We have seen from Example 1 that changing the order of two quantifiers can sometimes change the meaning of a statement. However, if the quantifiers are both $\forall$ or both $\exists$, then the order can always be switched without affecting the meaning of the statement.

Example 8 *Analyze the logical forms of the following statements.*

1. *All friends like each other.*

2. *Everyone likes at least two people.*

3. *James likes exactly one person.*

In general, for any set A, statement $\forall x \in AP(x)$ means that for every value of x in the set A, $P(x)$ is true. Statement $\exists x \in AP(x)$ means that there is some value of x that is in A and that also makes $P(x)$ to be true. Therefore, $\forall x \in AP(x)$ is equivalent to $\forall x(x \in A \rightarrow P(x))$, and $\exists x \in AP(x)$ is equivalent to $\exists x(x \in A \land P(x))$.

Formal Logical Notation

"$\forall x$ in $A, P(x)$"	can be written as	"$\forall x(x \in A \rightarrow P(x))$"
"$\exists x$ in A such that $P(x)$"	can be written as	"$\exists x(x \in A \land P(x))$"

Note that if $A = \varnothing$, then $\exists x \in AP(x)$ is false no matter what P stands for. What about $\forall x \in AP(x)$? Is it true?

$$\begin{array}{lll} & \forall x \in AP(x) & \\ \text{is equivalent to} & \neg\neg\forall x \in AP(x) & \text{Double Negation law} \\ \text{is equivalent to} & \neg\exists x \in A\neg P(x) & \text{by Quantifier Negation law} \end{array}$$

If $A = \varnothing$, then $\exists x \in A\neg P(x)$ must be false, and so $\neg\exists x \in A\neg P(x)$ must be true. Therefore, $\forall x \in AP(x)$ must true. Also note that since $\forall x \in AP(x)$ is equivalent to $\forall x(x \in A \to P(x))$, if $A = \varnothing$, $x \in A$ is always false, and so the implication is always true. Such a statement is often said *vacuously* true.

As another note: universal quantifier distributes over conjunction. That is, $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall xP(x) \wedge \forall xQ(x)$. Why? However, existential quantifier does not distributes over conjunction. In other words, $\exists x(P(x) \wedge Q(x))$ is not equivalent to $\exists xP(x) \wedge \exists xQ(x)$. Why?

As an example of the distributive law for the universal quantifier and conjunction, let's consider the equation $A = B$ where A and B are two sets. The equation $A = B$ means $\forall x(x \in A \leftrightarrow x \in B)$, and so

$$\begin{array}{lcl} & \forall x(x \in A \leftrightarrow x \in B) & \\ \text{is equivalent to} & \forall x[(x \in A \to x \in B) \wedge (x \in B \to x \in A)] & \text{Def of Biconditional} \\ \text{is equivalent to} & \forall x(x \in A \to x \in B) \wedge \forall x(x \in B \to x \in A) & \text{Distributive law} \\ \text{is equivalent to} & A \subseteq B \wedge B \subseteq A & \text{Def of subset} \end{array}$$

Example 9 *Analyze the logical forms of the following statements where the universe of discourse is* $\mathbb{N}$.

1. x *is a perfect square.*
2. x *is the smallest number that is a multiple of both* y *and* z.

Exercise 2 *Analyze the logical form of the statement "Every positive number has exactly two square roots". The universe of discourse is* $\mathbb{R}$.

Exercise 3 *Show that the statements* $A \subseteq B$ *and* $A \setminus B = \varnothing$ *are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.*

3 More on Sets

3.1 Elementhood Test Notation and Logical Statements

We have seen the elementhood test notation for defining a set. For example, the set of all perfect squares can be written as $S = \{n^2 \mid n \in \mathbb{N}\}$. This also can be written as $\{x \mid \exists n \in \mathbb{N}(x = n^2)\}$. Therefore, the logical statement $x \in \{n^2 \mid n \in \mathbb{N}\}$ means the same thing as $\exists n \in \mathbb{N}(x = n^2)$.

Example 10 *Analyze the logical forms of the following statements.*

1. $y \in \{\sqrt[3]{x} \mid x \in \mathbb{Q}\}$.
2. $\{n^2 \mid n \in \mathbb{N}\}$ *and* $\{n^3 \mid n \in \mathbb{N}\}$ *are not disjoint.*

3.2 Power Set

Definition 2 (power set) *Suppose that A is a set. The power set of A, denoted $\mathcal{P}(A)$, is the set whose elements are all the subsets of A. In other words, $\mathcal{P}(A) = \{x \mid x \subseteq A\}$.*

For example, the set $A = \{1, 2\}$ has four subsets: $\varnothing$, $\{1\}, \{2\}$, and $\{1, 2\}$. Therefore, $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.
Why is $\varnothing$ a subset of any set?
What is $\mathcal{P}(\varnothing)$?

Example 11 *Analyze the logical forms of the following statements.*

1. $x \in \mathcal{P}(A)$.
2. $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
3. $x \in \mathcal{P}(A \cap B)$.
4. $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

3.3 Cartesian Products

Definition 3 *Let n be a positive integer and let $x_1, x_2, ..., x_n$ be (not necessarily distinct) elements. The* ordered n-tuple, *$(x_1, x_2, ..., x_n)$, consists of $x_1, x_2, ..., x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an* ordered pair, *and an ordered 3-tuple is called an* ordered triple.

Two ordered n-tuple $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ are equal *if, and only if, $x_1 = y_1, x_2 = y_2, ..., x_n = y_n$.*

Symbolically:

$$(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n) \leftrightarrow x_1 = y_1, x_2 = y_2, ..., x_n = y_n.$$

In particular, $(a, b) = (c, d) \leftrightarrow a = c$ and $b = d$.

For example, $(1, 2, 3, 4) \neq (1, 3, 2, 4)$ and $(3, (-2)^2, \frac{1}{2}) = (\sqrt{9}, 4, \frac{3}{6})$.

Definition 4 *Given sets $A_1, A_2, ..., A_n$, the Cartesian product of $A_1, A_2, ..., A_n$ denoted $A_1 \times A_2 \times ... \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, ..., a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, ..., a_n \in A_n$.*

Symbolically:

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) \mid a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$$

is the Cartesian product of A1 and A2.

Example 12 *Let $A_1 = \{x, y\}, A_2 = \{1, 2, 3\}, and A_3 = \{a, b\}$.*

1. *Find $A_1 \times A_2$.*
2. *Find $(A_1 \times A_2) \times A_3$.*
3. *Find $A_1 \times A_2 \times A_3$.*

4 Set Identities

Recall that in Propositional Logic, we have the following logical equivalences.
Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation laws:	$p \vee \neg p \equiv \mathbf{t}$	$p \wedge \neg p \equiv \mathbf{c}$
Double negative law:	$\neg(\neg p) \equiv p$	
Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound laws:	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of $\mathbf{t}$ and $\mathbf{c}$:	$\neg \mathbf{t} \equiv \mathbf{c}$	$\neg \mathbf{t} \equiv \mathbf{c}$

In Set Theory, we have *set identities* which are equations universally true for all elements in some set.

Theorem 1 (Set Identities) *Let the universal set be U. For every set $A \subseteq U$, set $B \subseteq U$, and set $C \subseteq U$,*

Commutative laws:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative laws:	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	$A \cup \varnothing = A$	$A \cap U = A$
Complement laws:	$A \cup A^c = U$	$A \cap A^c = \varnothing$
Double Complement law:	$(A^c)^c = A$	
Idempotent laws:	$A \cup A = A$	$A \cap A = A$
Universal bound laws:	$A \cup U = U$	$A \cap \varnothing = \varnothing$
De Morgan's laws:	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
Absorption laws:	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complements of U and $\varnothing$:	$U^c = \varnothing$	$\varnothing^c = U$
Set Difference Law:	$A \setminus B = A \cap B^c$	

With set identities, we can prove some set properties algebraically (Section 6.3).

Example 13 *Construct an algebraic proof that for all sets A and B, $A \setminus (A \cap B) = A \setminus B$.*

5 Proof Involving Quantifiers

How to prove a goal of the form $\forall x P(x)$? If your proof works no matter what x was, then you can conclude that $\forall x P(x)$ is true. This means that your proof should work for any value of x. Thus, you should start your proof without any assumption about x.
Strategy 1 (for all): (a) To prove a goal of the form $\forall x P(x)$, let x be an arbitrary object and prove $P(x)$. The letter x must be a new variable in the proof. If x is already being used in the proof to stand for something, then you must choose an unused variable, say y, to stand for the arbitrary object, and prove $P(y)$. (b) To use a given assumption of the form $\forall x P(x)$, plug in any value, say m, for x and use this assumption to conclude that $P(m)$ is true. This rule is called *universal instantiation*.

Form of the Proof
Let x be arbitrary.
[Proof of $P(x)$ goes here]
Since x was arbitrary, we can conclude that $\forall x P(x)$.

Example 14 *Suppose A, B, and C are sets, and $A \setminus B \subseteq C$. Prove that $A \setminus C \subseteq B$.*
Proof sketch.
Given: $A \setminus B \subseteq C$
Goal: $A \setminus C \subseteq B$
Analyze the logical form of the goal: $\forall x(x \in A \setminus C \to x \in B)$
Let x be arbitrary.
New Goal: $x \in A \setminus C \to x \in B$

Let x be arbitrary.
[Proof of $x \in A \setminus C \to x \in B$ goes here]
Since x was arbitrary, we can conclude that $\forall x(x \in A \setminus C \to x \in B)$, so $A \setminus C \subseteq B$.

The main advantage of using this strategy to prove a goal of the form $\forall x P(x)$ is to prove a goal about all objects by reasoning about only one object, as long as that object is arbitrarily chosen.

Example 15 *Suppose A and B are sets. Prove that if $A \cap B = A$ then $A \subseteq B$.*
Proof sketch.
Given: $A \cap B = A$
Goal: $\forall x(x \in A \to x \in B)$
Given: $A \cap B = A$, $x \in A$
Goal: $x \in B$

Suppose that $A \cap B = A$.
Let x be arbitrary.
Suppose that $x \in A$.
[Proof of $x \in B$ goes here]
Therefore $x \in A \to x \in B$.
Since x was arbitrary, we can conclude that $\forall x(x \in A \to x \in B)$, so $A \subseteq B$.
Therefore, if $A \cap B = A$, then $A \subseteq B$.

When we write up the final proof we can skip some or all of the last three sentences.

Strategy 2 (exists): (a) To prove a goal of the form $\exists x P(x)$, we need to find a value of x where $P(x)$ is true. Let x be the value we decide and show that $P(x)$ is true for this value. Same as in the case of $\forall x P(x)$, x should be a new variable. If necessary, rename x into some unused variable, say y, and rewrite the goal in the equivalent form $\exists y P(y)$. (b) To use a given assumption if the form $\exists x P(x)$, we introduce a new variable, say x_0, to stand for an object for which $P(x_0)$ is true. Now assume that $P(x_0)$ is true and use this as an assumption. This rule is called *existential instantiation.*

Form of the Proof
Let $x =$ the value we decide. .
[Proof of $P(x)$ goes here]
Therefore, $\exists x P(x)$.

Example 16 *Prove that for every real number x, if $x > 0$ then there is a real number y such that $y(y+1) = x$.*
Proof sketch.
Goal: $\forall x(x > 0 \to \exists y[y(y+1) = x])$
Let x be an arbitrary number
Given: $x > 0$
Goal: $\exists y[y(y+1) = x]$
We need to find a value of y such that $y(y+1) = x$.
Solve the quadratic equation:

Recall that the roots of a quadratic equation $ax^2 + bx + c = 0$ can be obtained by using Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Definition 5 (Divisible) *If n and d are integers and $d \neq 0$ then*
n is divisible by *d if, and only if, n equals d times some integer.*
Instead of "n is divisible by d", we can say that
n is a multiple of d, *or*
d is a factor of n, *or*
d is a divisor of n, *or*
d divides n.
The notation $d|n$ is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$:

$$d|n \textit{ iff } \exists k \in \mathbb{Z} \textit{ such that } n = dk.$$

For example, $4|20$, since $5 \times 4 = 20$.
Remark. When does d not divide n? An integer d does not divide n, denoted by $d \not| n$, if and only if, for every integer k, $n \neq dk$, or, in other words, the quotient n/d is not an integer.
Does k divides 0 where k is any nonzero integer?

Example 17 (Transitivity of Divisibility) *For all integers a, b, and c, if $a|b$ and $b|c$, then $a|c$.*

Definition 6 (Prime and Composite) *An integer n is prime if, and only if, $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n. An integer n is composite if, and only if, $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.*

Is every integer greater than 1 either prime or composite?

Example 18 (Divisibility by a Prime) *Prove that any integer $n > 1$ is divisible by a prime number.*

Example 19 *Is the following statement true or false? For all integers a and b, if $a|b$ and $b|a$ then $a = b$.*

Theorem 2 (Unique Factorization of Integers Theorem (Fundamental Theorem of Arithmetic)) *Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers $p_1, p_2, ..., p_k$, and positive integers $e_1, e_2, ..., e_k$ such that*

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

and any other expression for n as a product of prime numbers is identical to this except, perhaps, for the order in which the factors are written.

Definition 7 *Given any integer $n > 1$, the* standard factored form *of n is an expression of the form*

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

where k is a positive integer; $p_1, p_2, ..., p_k$ are prime numbers; $e_1, e_2, ..., e_k$ are positive integers; and $p_1 < p_2 < \cdots < p_k$.

Example 20 *Suppose m is an integer such that*

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot m = 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10.$$

Does $17|m$?

Theorem 3 (The Quotient-Remainder Theorem) *Given any integer n and positive integer d, there exist unique integers q and r such that*

$$n = dq + r \text{ and } 0 \leq r < d.$$

Definition 8 (div and mod) *Given an integer n and a positive integer d,*

$$\begin{aligned} n \ \textbf{\textit{div}}\ d &= \text{the integer quotient obtained when } n \text{ is divided by } d \\ n \ \textbf{\textit{mod}}\ d &= \text{the nonnegative integer remainder obtained when } n \text{ is divided by } d. \end{aligned}$$

Symbolically, if n and d are integers and $d > 0$, then

$$n \ div\ d = q \text{ and } n \ mod\ d = r \leftrightarrow n = dq + r$$

where q and r are integers and $0 \leq r < d$.

Definition 9 (Parity) *The* parity *of an integer refers to whether the integer is even or odd. We call the fact that any integer is either even or odd the* parity property*.*

Example 21 *Prove that the square of any odd integer has the form $8m + 1$ for some integer m.*

Definition 10 (mutually disjoint) *Sets A_1, A_2, A_3, ... are* mutually disjoint *(or* pairwise disjoint *or* nonoverlapping*) if, and only if, no two sets A_i and A_j have any elements in common, where $i \neq j$. Symbolically, for all $i, j = 1, 2, 3, ...$, $A_i \cap A_j = \varnothing$ whenever $i \neq j$.*

Definition 11 (partition) *A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 ...\}$ is a partition of a set A if, and only if,*

1. *A is the union of all the A_i*
2. *The set $A_1, A_2, A_3, ...$ are mutually disjoint.*

Example 22 *Let $\mathbb{Z}$ be the set of all integers and let*

- $T_0 = \{n \in \mathbb{Z} \,|\, n = 3k, \text{ for some integer } k\}$,
- $T_1 = \{n \in \mathbb{Z} \,|\, n = 3k + 1, \text{ for some integer } k\}$, *and*
- $T_2 = \{n \in \mathbb{Z} \,|\, n = 3k + 2, \text{ for some integer } k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Definition 12 *For any real number x, the* absolute value *of x, denoted $|x|$, is defined as follows:*
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Lemma 1 *For all real numbers r, $-|r| \leq r \leq |r|$.*

Lemma 2 *For all real numbers r, $|-r| = |r|$.*

Example 23 (The Triangle Inequality) *Prove that for all real numbers x and y, $|x+y| \leq |x| + |y|$.*

More examples on how to apply these strategies will be given when we learn predicate logic.

Sometimes when proving a biconditional $P \leftrightarrow Q$, the steps for proving $P \to Q$ is same as the steps for proving $Q \to P$ in a reversed order. In this case, we could simplify the proof by writing it as a string of equivalences, starting with P and ending with Q.

Example 24 *Suppose A, B, and C are sets. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus C$.*
Proof Sketch.
$x \in A \cap (B \setminus C)$ *iff* $x \in A \wedge x \in B \setminus C$ *iff* $x \in A \wedge x \in B \wedge x \notin C$;
$x \in (A \cap B) \setminus C$ *iff* $x \in A \cap B \wedge x \notin C$ *iff* $x \in A \wedge x \in B \wedge x \notin C$.

Proof by cases is one of the strategies we discussed before when a given asumption is of the form $P \vee Q$.

Example 25 *Suppose that A, B, and C are sets. Prove that if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.*
Proof Sketch.
Given: $A \subseteq C$, $B \subseteq C$
Goal: $\forall x (x \in A \cup B \to x \in C)$
Let x be arbitrary and assume that $x \in A \cup B$
Given: $A \subseteq C$, $B \subseteq C$, $x \in A \vee x \in B$
Goal: $x \in C$

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- *Discrete Mathematics with Applications, 4th Edition* by Susanna Epp, Cengage Learning 2010.
- *How to Prove It: A Structured Approach, 2nd Edition* by Daniel J. Velleman, Cambridge University Press 2006