Conditional Probability

CS231
Dianna Xu
Boy or Girl?

• A couple has two children, one of them is a girl. What is the probability that the other one is also a girl? Assuming 50/50 chances of conceiving boys and girls.
Conditional Probability

• Let $A$ and $B$ be events in a sample space $S$. If $P(A) \neq 0$, then the conditional probability of $B$ given $A$, $P(B|A)$ is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A) = \frac{P(A \cap B)}{P(B|A)}$$

$$P(A \cap B) = P(B|A) \times P(A)$$
Example

• Two cards are drawn from a well-shuffled deck. What is the probability that:
  – both are kings?
  – second draw is a king?

• $A = \text{1st draw is king}, \ B = \text{2nd draw is king}$
• $P(A) = 4/52, \ P(\bar{A}) = 48/52$
• $P(B|A) = 3/51, \ P(B|\bar{A}) = 4/51$
• $P(A \cap B) = 4/52 \times 3/51 = 12/2652$
• $P(A \cap B) + P(\bar{A} \cap B) = 4/52 \times 3/51 + 48/52 \times 4/51$
Example

• If the experiment of drawing a pair is repeated over time, what would be the expected value of the number of kings?
• 2 kings: \( P(A \cap B) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} \)
• 1 king: \( P(\bar{A} \cap B) + P(A \cap \bar{B}) = \frac{48}{52} \times \frac{4}{51} + \frac{4}{52} \times \frac{48}{51} = \frac{384}{2652} \)
• Expected value of # of kings: \( 2 \times \frac{12}{2652} + 1 \times \frac{384}{2652} \approx 0.154 \)
Example

• 5% of manufactured components are defective in general.
• The method for screening out defective items is not totally reliable. The test rejects good parts as defective in 1% of the cases and accepts defective parts as good ones in 10% of the cases.
• Given that the test indicates that an item is good, what is the probability that this item is, in fact, defective?
Definitions

- $T = \text{A component cleared the test (tested good)}$
- $D = \text{A component is defective}$
- $\bar{T} = \text{A component did not clear the test (tested defective)}$
- $G = \text{A component is good (} G = \bar{D} \text{)}$

- Want to solve: $P(D | T)$
\[ P(D|T) \]

- \( P(D) = 0.05, \ P(G) = 0.95 \)
- \( P(\bar{T}|G) = 0.01 \) (false positive) \( P(T|G) = 0.99 \)
- \( P(T \cap G) = P(T|G) \times P(G) = 0.99 \times 0.95 = 0.9405 \)
- \( P(T|D) = 0.1 \) (false negative)
- \( P(T \cap D) = P(T|D) \times P(D) = 0.1 \times 0.05 = 0.005 \)

- \( T = (T \cap G) \cup (T \cap D) \)
- \( P(T) = 0.9405 + 0.005 = 0.9455 \)
- \( P(D|T) = \frac{P(T \cap D)}{P(T)} = \frac{0.005}{0.9455} = 0.0052882 \)
Medical Screening

• 1% of population suffer from a certain disease.
• The method for screening is not totally reliable. The test reports false positive in 5% of the cases and false negative in 10% of the cases.
• Given that a person has a negative test result, what is the probability that this person is, in fact, sick?
• Given that a person has a positive test result, what is the probability that this person is, in fact, sick?
Definitions

- \( T \): A person cleared the test (negative)
- \( S \): A person is sick
- \( \bar{T} \): A person did not clear the test (positive)
- \( H \): A person is healthy \( (H = \bar{S}) \)

Want to solve: \( P(S|T) \) and \( P(S|\bar{T}) \)
\[ P(S|T) \]

- \( P(S) = 0.01 \), \( P(H) = 0.99 \)
- \( P(\bar{T}|H) = 0.05 \) (false positive) \( P(T|H) = 0.95 \)
- \( P(T \cap H) = P(T|H) \times P(H) = 0.95 \times 0.99 = 0.9405 \)
- \( P(T|S) = 0.1 \) (false negative)
- \( P(T \cap S) = P(T|S) \times P(S) = 0.1 \times 0.01 = 0.001 \)

- \( T = (T \cap H) \cup (T \cap S) \)
- \( P(T) = 0.9405 + 0.001 = 0.9415 \)
- \( P(S|T) = \frac{P(T \cap S)}{P(T)} = \frac{0.001}{0.9415} \approx 0.001 \)
Bayes’ Theorem

• If \( E_1, E_2, \ldots, E_n \) are mutually exclusive and exhaustive events in a sample space, the total probability of any event \( F \) is:

\[
P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)
\]

• For any event \( E \) and \( F \) with \( P(F) \neq 0 \):

\[
P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|\bar{E})P(\bar{E})}
\]
$P(S|T)$ and $P(S|\bar{T})$ with Bayes’

- $P(S) = 0.01$, $P(H) = P(\bar{S}) = 0.99$
- $P(\bar{T}|H) = P(\bar{T}|\bar{S}) = 0.05$, $P(T|H) = P(T|\bar{S}) = 0.95$
- $P(T|S) = 0.1$, $P(\bar{T}|S) = 0.9$

\[
P(S|T) = \frac{P(T|S)P(S)}{P(T|S)P(S) + P(T|\bar{S})P(\bar{S})}
\]
\[
= \frac{0.1\times0.01}{0.1\times0.01 + 0.95\times0.99} = \frac{0.001}{0.001 + 0.9405} \approx 0.001
\]

\[
P(S|\bar{T}) = \frac{P(\bar{T}|S)P(S)}{P(\bar{T}|S)P(S) + P(\bar{T}|\bar{S})P(\bar{S})}
\]
\[
= \frac{0.9\times0.01}{0.9\times0.01 + 0.05\times0.99} = \frac{0.009}{0.009 + 0.0495} \approx 0.1538
\]
Bayes’ Theorem

• If \( E_1, E_2, \ldots, E_n \), are mutually exclusive and exhaustive events in a sample space, given any \( F \) with \( P(F) \neq 0 \),

\[
P(E_k | F) = \frac{P(F | E_k)P(E_k)}{\sum_{i=1}^{n} P(F | E_i)P(E_i)} = \frac{P(F | E_k)P(E_k)}{P(F)}
\]

• \( P(\theta) \) is also known as prior

• \( P(\theta | X) \) is the posterior probability after observing \( X \) and obtaining \( P(X | \theta) \)
Monty Hall Revisited

- $A/B/C = \text{prize is behind door } A/B/C$
- $M_A/M_B/M_C = \text{Monty opens door } A/B/C$
- You choose door A and Monty opens a door revealing no prize, say door B
- $P(M_B|A) = 1/2, \ P(M_B|B) = 0, \ P(M_B|C) = 1$
- $P(M_B) = 1/2, \ P(A) = P(B) = P(C) = 1/3$
- $P(A|M_B) = \frac{P(M_B|A)P(A)}{P(M_B)} = 1/2 \times 1/3 / 1/2 = 1/3$
- $P(C|M_B) = \frac{P(M_B|C)P(C)}{P(M_B)} = 1 \times 1/3 / 1/2 = 2/3$
- Exact same analysis holds for $M_C$
Bayes’ Ratio

• When there are three events, $A$, $B$ and $C$ and the comparative posterior probabilities are of interest, consider the ratio:

$$\frac{P(A|C)}{P(B|C)} = \frac{P(C|A)}{P(C|B)} \times \frac{P(A)}{P(B)}$$
Example

• Two bags, one contains 70 red and 30 blue balls, and the other 30 red and 70 blue balls.

• Choose one bag randomly and draw with replacement.

• 8 red and 4 blue balls are drawn in 12 tries.

• What is the probability that it was the predominantly red bag that was chosen?
Solution

- $A$ = selecting the 1st bag, $B$ = selecting the 2nd bag, $C$ = getting the draws we did
- $P(C|A) = (\frac{7}{10})^8 \times (\frac{3}{10})^4 \times C(12,8)$
- $P(C|B) = (\frac{7}{10})^4 \times (\frac{3}{10})^8 \times C(12,8)$
- $P(A) = P(B) = 0.5$

\[
\frac{P(C|A)}{P(C|B)} = \left(\frac{7}{10}\right)^4 \div \left(\frac{3}{10}\right)^4 = \left(\frac{7}{3}\right)^4
\]

\[
\frac{P(A|C)}{P(B|C)} = \left(\frac{7}{3}\right)^4 \Rightarrow \left\{\begin{array}{l}
P(A|C) = \frac{\left(\frac{7}{3}\right)^4}{\left(\frac{7}{3}\right)^4 + 1} \\
P(A|C) + P(B|C) = 1
\end{array}\right.
\]
Dramatic Taxicab

• A cab was involved in a hit-and-run at night.
• Two cab companies operate in the city, with green and blue cabs, respectively.
• 85% of the cabs are green.
• A witness identified the cab as blue.
• The witness correctly identified the two colors 80% of the time under night-time testing.
• What is the probability that the witness was right?
Independent Events

• Two events are independent when the occurrence of one does not affect the probability of the other.
  – tossing coins
  – rolling dice

• Events $A$ and $B$ are independent iff:
  \[ P(A \cap B) = P(A) \times P(B) \]
\[ P(A \cap \overline{B}) \]

- If \( A \) and \( B \) are independent events, so are \( A \) and \( \overline{B} \).

- From set theory:
  - \((A \cap B) \cup (A \cap \overline{B}) = A\)
  - \((A \cap B) \cap (A \cap \overline{B}) = \emptyset\)

- \( P\left((A \cap B) \cup (A \cap \overline{B})\right) = P(A \cap B) + P(A \cap \overline{B}) = P(A) \)

- \( P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(\overline{B}) \)
Loaded Coin

• A coin is loaded so that the probability of heads is 0.6. After 10 tosses, what is the probability of obtaining 8 heads?

• Consider HHHHHHHHHTT

• $P(HHHHHHHHHTT) = 0.6^8 \times 0.4^2$

• How many ways can you get 8 heads with 10 tosses? – $C(10, 8)$

• $P(8 \text{ heads}) = C(10, 8) \times 0.6^8 \times 0.4^2 \approx 0.12$