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Math/CS 231 Exam 2 11/18/09

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1. Short Answers

   (a) What is gcd(120, 35)? Show the steps of the gcd algorithm application.

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\begin{aligned}
120 = 35 * 3 + 15 &\Rightarrow gcd(120, 35) = gcd(35, 15) \\
35 = 15 * 2 + 5 &\Rightarrow gcd(35, 15) = gcd(15, 5) \\
15 = 5 * 3 + 0 &\Rightarrow gcd(15, 5) = gcd(5, 0) \\
& gcd(120, 35) = gcd(5, 0) = 5
\end{aligned}
$$

   (b) What is the difference between ordinary and strong mathematical induction?

   Ordinary mathematical induction proves a single base case of P(a) is true. The inductive hypothesis assumes P(k) $(k \geq a)$ is true and then proves P(k+1) is also true.

   Strong mathematical induction proves possibly mutiple base cases P(a), P(a+1), $\cdots$, P(b) are true. The inductive hypothesis assumes that P(i) is true $\forall i, b < i \leq k$, then proves P(k+1) is also true.

   (c) Write the negation of the statement $\exists x \in U(x \in A \cap B \rightarrow x \notin C)$.

   $\forall x \in U, x \in A \cap B \wedge x \in C$

   (d) State the Well Ordering Principle for integers.

   Given a non-empty set S of integers, all of which are greater than some fixed integer, S has a least element.

2. Let $A = \{a, b, c, d, e\}, B = \{1, 2, a, 3, c\}$ and $C = \{2, b, a, 1\}$. Find each of the following:

   (a) $A \backslash C$

   $A \backslash C = \{c, d, e\}$

   (b) $A \cap (B \cup C)$

   $A \cap (B \cup C) = A \cap \{1, 2, a, 3, c, b\} = \{a, b, c\}$

(c) $(A \cap B) \times (A \cap C)$ (Cartesian product)
$(A \cap B) \times (A \cap C) = \{a, c\} \times \{a, b\} = \{(a, a), (a, b), (c, a), (c, b)\}$

3. Given any two arbitrary sets A and B, show that $A^c \cap (A \cup B) = B \backslash A$.

   Algbraic proof: $A^c \cap (A \cup B) = (A^c \cap A) \cup (A^c \cap B) = \emptyset \cup (A^c \cap B) = A^c \cap B = B \setminus A$.

   Alternatively: show that $A^c \cap (A \cup B) \subseteq A \setminus B$ and $A \setminus B \subseteq A^c \cap (A \cup B)$

   $A^c \cap (A \cup B) \subseteq A \setminus B$: Let x be an arbitrary element in $A^c \cap (A \cup B)$, by definition, $x \in A^c$ and $x \in (A \cup B)$, which is equivalent to $x \notin A$ and ($x \in A$ or $x \in B$), and it follows that $x \notin A$ and $x \in B$, which is by definition $x \in B \setminus A$

   $A \setminus B \subseteq A^c \cap (A \cup B)$: Let x be an arbitrary element in $A \setminus B$. by definition, $x \in B$ and $x \notin A$. $x \in B \rightarrow x \in (A \cup B)$ and $x \notin A \rightarrow x \in A^c$. Thus $x \in (A \cup B)$ and $x \in A^c$, by definition, $x \in A^c \cap (A \cup B)$.

4. Prove that if n is an integer and $3n + 1$ is odd, then n is even. You need to do this proof in two ways (out of three): direct proof, proof by contrapositive, or proof by contradiction. Each proof is worth the same amount.

   Proof Method 1 (circle one): Direct proof   Proof by contrapositive)   Proof by contradiction

   Direct proof: Let n be an integer, and assume $3n + 1$ is odd. Then there exists an integer j such that $3n + 1 = 2j + 1$. Thus $3n = 2j$, so that $3n$ is even. But then n is even, for if n were odd, then $3n$ would be odd.

   Proof by contrapositive: Assume that n is odd. Then there exists an integer j such that $n = 2j + 1$. Then $3n + 1 = 3(2j + 1) + 1 = 6j + 4 = 2(3j + 2)$, so $3n + 1$ is even.

   Proof by contradiction: Assume the statement is false. Then there exists an integer n such that $3n+1$ is odd and n is odd. Then $n = 2j + 1$ for some integer j, so $3n + 1 = 2(3j + 2)$, so that $3n + 1$ is even. But $3n + 1$ is odd, a contradiction. Therefore the original statement is true.

5. Using induction, prove that for all integers $n \geq 1$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}.$$

   For $n = 1$, $1 \cdot 2 = 1(2)(3)/3 = 2$, so $P(1)$ is true.

   Assume that $P(k)$ is true for $k \geq 1$. Then $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1) + (k + 1)(k + 2) = \frac{k(k+1)(k+2)}{3} + (k + 1)(k + 2) = \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$, so $P(k + 1)$ is true. The statement is true for all $n \geq 1$ by mathematical induction.

6. Do ONE of the following two problems ((a) OR (b)):

   (a) Prove the correctness of the following algorithm:

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[Pre-condition: i=1 and sum=0]
while(i<=100)
  sum := sum + i
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  i := i + 1
end while
[Post-condition: sum = 1+ 2 + ... + 100]
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Loop invariant: I(n): i = n and sum = 1 + ... + n

There is a typo in the loop invariant, it should be: I(n): i = n+1 and sum = 1 + ... + n

i. Base case: I(0): i=1, since n=0, sum=0, which matches the pre-condition

ii. Inductive: Assume that before an arbitrary iteration k, I(k-1) is true, i.e. i = k and sum = 1 + ... + k-1
loop iteration execution:
`sum := sum + i` $\Rightarrow$ sum = 1 + ... + (k-1) + k
`i := i + 1` $\Rightarrow$ i = k+1
Thus I(k) is true after one loop iteration

iii. Eventual falsity of guard: i starts at 1 and is incremented at each iteration until $\leq$ 100 is violated, which is at I(100).

iv. Correctness of post-condition: I(100): i = 101 and sum = 1+ ... + 100, which matches the post-condition.

(b) Prove by induction that for all integers $n \geq 4$,

$$2n + 3 \leq 2^n.$$

For $n = 4, 2 \cdot 4 + 3 = 11 \leq 2^4 = 16$, so $P(4)$ is true. Assume $P(k)$ is true for $k \geq 4$, so that $2k + 3 \leq 2^k$. Then $2(k + 1) + 3 = 2k + 2 + 3 = 2k + 3 + 2 \leq 2^k + 2 \leq 2^k + 2^k = 2^{k+1}$, so that $P(k + 1)$ is true. Therefore the statement is true for all $n \geq 4$.