Name: Solutions

Math/CS 231 Practice Proofs 11/13/09

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1. Prove that if m is an even integer, then $m + 7$ is an odd integer. You need to do this proof in **two ways** (out of three): direct proof, proof by contrapositive, or proof by contradiction. Each proof is worth the same amount.

   Proof Method 1 Direct proof

   Proof. Assume that m is an even integer. Then there exists an integer k such that $m = 2k$. Then $m + 7 = 2k + 7 = 2k + 6 + 1 = 2k + 2 \cdot 3 + 1 = 2(k + 3) + 1$. Since k and 3 are integers, $k + 3$ is an integer, so $2(k + 3)$ is an even integer. Hence $m + 7 = 2(k + 3) + 1$ is an odd integer.

   Proof Method 2 Proof by contrapositive

   We will prove that if $m + 7$ is an even integer, then m is an odd integer.

   Proof. If $m + 7$ is even, then there exists an integer k such that $m + 7 = 2k$. Thus $m = 2k - 7 = 2k - 2 \cdot 3 - 1 = 2(k - 3) - 1$. Since k and 3 are integers, $k - 3$ is an integer, so $2(k - 3)$ is an even integer. Hence $m = 2(k - 3) - 1$ is an odd integer.

   Proof Method 3 Proof by contradiction

   Proof. Assume there exists an integer m such that m is even and $m + 7$ is even. Then there exist integers k and j such that $m = 2k$ and $m+7 = 2j$. Thus $m = 2k$ and $m = 2j-7$, so that $2k = 2j-7$, or $2(k - j) = -7$. Since k and j are integers, $k - j$ is an integer, thus -7 is an even integer. However, $-7 = 2 \cdot (-3) - 1$, so -7 s an odd integer, a contradiction. Thus the assumption is false, and the original statement is true.

2. Using induction, prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3; i.e. $3 \mid 2^{2n} - 1$.

   Base case: n = 1
   $2^{2*1} - 1 = 3$, and $3 \mid 3$.
   Inductive hypothesis: assume n = k is true
   $3 \mid 2^{2k} - 1 \Longrightarrow 2^{2k} - 1 = 3 * r, r \in \mathcal{Z}$.
   Prove for n = k+1:

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\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= 2^{2k} * 4 - 1 \\
&= 2^{2k} * 3 + 2^{2k} - 1 \\
&= 2^{2k} * 3 + 3 * r \ (\textit{inductive hypothesis}) \\
&= (2^{2k} + r) * 3
\end{aligned}
$$

   $r \in \mathcal{Z}$, and $2^{2k} \in \mathcal{Z} \Longrightarrow (2^{2k} + r) \in \mathcal{Z}$, thus $3 \mid 2^{2(k+1)} - 1$ $\square$

3. Given sets A, B, and C in the same universe, determine if each of the following statements is true or false. If it is true, then prove it. If it is false, then give a counter example.

   (a) $C \subseteq A$ and $C \subseteq B \rightarrow C \subseteq A \cup B$
   Proof. Let $x \in C$. Then since $C \subseteq A$ and $C \subseteq B$, $x \in A$ and $x \in B$, so that $x \in A \cap B$. Thus $x \in A \cup B$. The statement is True.

(b) $C \subseteq A \cup B \rightarrow C \subseteq A$ and $C \subseteq B$

The statement is False. A counterexample is given by $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{2, 3\}$.