Introduction to the Analysis of Algorithms

Based on the notes from David Fernandez-Baca

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CS206 Intro to Data Structures

Algorithm

- An *algorithm* is a strategy (well-defined computational procedure) for solving a problem, independent of the actual implementation.

ARRAY EQUALITY

Input: Two arrays A and B, of the same length and without duplicates.
Question: Do A and B contain the same elements?

Problem: Array Equality

Algorithm 1

```
for each position i in array A
    if element A[i] does not appear in array B
        return false
return true
```

Algorithm 2

```
Make a copy of both arrays and sort them
for each position i
    if A[i] is different from B[i]
        return false
return true
```

Measurement of a Better Strategy

Which strategy is better? Some potential considerations are:

- Speed?
- Memory consumption?
- Network bandwidth?
- Easiness of implementation?
- Reusability?

The most significant for us are the first two, and we will concentrate on the first one.

Time Complexity

- The **time complexity** (or **running time**) of an algorithm is a function that describes the number of basic execution steps in terms of the *input size*.
- The time complexity abstracts the components of an algorithm's performance that depend on the algorithm itself away from those components that are machine- and implementation-dependent.

Example: Sequential Search

SEARCH

Input: An array A of length n and a value v
Problem: Determine whether A contains v.

i = 0;	**assignment**: 1 step
while i < n	**test**: n + 1 steps
if A[i]==v	**test**: 1 step * n iterations of the loop
return true	**return***:* 1 step (only once!)
i++	**increment**: n times
return false	**return**: 1 step (only once!)

Example: Sequential Search

- For the worst case, the total number of steps is $T(n) = 3n + 3$.
- The execution time for an input of length n is **proportional to** $T(n)$.
- As n gets larger, the extra "+3" becomes relatively insignificant, so the execution time is roughly proportional to 3n.
- We can simplify this statement further and say that $T(n)$ is proportional to n or **linear** in n: $f(n) = n$.
- **Worst-case time complexity** of this algorithm is $O(n)$, or "big-O of n".

Asymptotic upper bound: *O*-notation

Definition:

$T(n)$ is $O(f(n))$ if and only if there exist positive constants c and N such that, for all $n \geq N$,

$$T(n) \leq c\, f(n)$$

$T(n)$ is $O(f(n))$ if you can multiply $f(n)$ by a (possibly large) constant (c) so that, **asymptotically** (as n shoots off to infinity), $T(n)$ is **completely underneath** $c\, f(n)$.

Example

Claim 1. $T(n) = 3n + 3$ is $O(n)$

Proof:

Choose $c=4$ and $N=3$. Then, for any $n \geq 3$,

$$3n + 3 \leq 3n + n \leq 4n$$

Claim 2. $T(n) = 42n + 17$ is $O(n)$

Proof:

Choose $c=43$ and $N=17$. Then, for any $n \geq 17$,

$$42n + 17 \leq 43n + n \leq 44n$$

General Principle

Fact 1. Every linear function $f(n) = an + b$ is $O(n)$.

Fact 2. When using-O notation we can ignore constant (multiplicative) factors!

Example: $T(n) = 109\, n + 109$ is $O(n)$.
Set $c = 2*109$ and $N = 1$.

You can think of $O(n)$ as the *class* of all functions that do not grow any faster than a linear function, at least for large values of n.

Array Equality, Revisited

Algorithm 1

```
for each position i in array A
    if element A[i] does not appear in array B
        return false
return true
```

For i = 0 to n-1, sequentially search for A[i] in array B.

Algorithm 1 Pseudocode

	#Times performed	
`i = 0`	1	
`while i < n`	n + 1	
`    found = false`	n	
`    j = 0`	n	
`    while j < n`	n × (n + 1)	*at most*
`        if a[i] == b[j]`	n × n	*at most*
`            found = true`	n × 1	
`            break`	n × 1	
`        ++j`	n × n	*at most*
`    if !found`	n	
`        return false`	0	
`    ++i`	n	
`return true`	1	
Total	$3n^2 + 8n + 3$	*at most*

Upper bound of Alg. 1

Claim 1. $T(n) = O(n^2)$

Proof:

Choose c=14(=3+8+3) and observe that as long as $n \geq 1$,

$$3n^2 + 8n + 3 \leq 3n^2 + 8n^2 + 3n^2 = 14n^2$$

- More generally, *every* quadratic function is $O(n^2)$.
- $O(n^2)$ is the class of all functions that asymptotically grow no faster than quadratic functions.
- Note that 3n+3 is also $O(n^2)$. However, we are most interested in describing an algorithm using the *smallest* (slowest growing) big-O class that we can identify. So, it is more precise to say that 3n+3 is O(n).
- Adding the extra constant-time steps does not add to the big-O complexity.

Array Operations

- Insertion
- Searching
- Deletion
- Display

- Ordered array:
  - int[] intArray = { 0, 3, 6, 9, 12, 15, 18, 21, 24, 27 };
- Unordered array:
  - int[] intArray = {18, 0, 3, 6, 24, 9, 12, 15, 21, 27 };

Complexity

- Linear search
  - O(N)
- Insertion in unordered array
  - O(1)
- Insertion in ordered array
  - O(N)
- Deletion in unordered array
  - O(N)
- Deletion in ordered array
  - O(N)

Binary Search (Ordered Arrays)

```
BinarySearch(A, v) // A must be sorted
    n = A.size
    left = 0
    right = n-1
    while left<=right
        mid = (left + right)/2
        if A[mid] == v
            return true
        if v < A[mid]
            right = mid -1
        else
            left = mid +1
    return false
```

Each iteration divides the search range [left..right] by 2.

When does the loop terminate?

- we find what we are looking for, or
- there are no more elements in the search range.

Thus, the number of iterations is bounded by the number of times we can divide n by 2 before we get 1. This number is known as the **log base 2 of n.**

Logarithms

```
int n = 32;
while (n > 1) { n = n/2; }
```

- 32= 2 * 2 * 2 * 2 * 2 = 2^5, it will take 5 iterations to get down to 1.
- The number 5 is called the *log base 2* of 32. It is the exponent x such that $2^x = 32$.
- For arbitrary n, the number of iterations equals the number x of times we can divide n by half so that we get 1.
- Thus, x is the exponent for which $n(1/2)^x = 1$. Equivalently, x is the number such that $2^x = n$; i.e.,x is the log base 2 of n.
- In general x will not be a whole number but is never more than 1 away from the number of iterations.

Subset Sum

SUBSET SUM

Input: An array A with n elements and a number K.
Question: Does A contain a subset elements that adds up to exactly K?

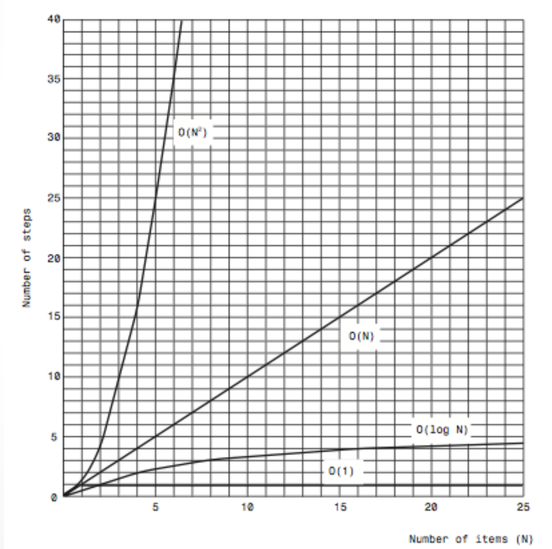
- Enumerate all subsets of the elements of A. For each subset, see if its elements add up to K.
- There are 2n subsets to enumerate. (Why?)
- Therefore, the algorithm takes $O(n*2^n)$ time.
- Subset Sum is *NP-complete*, which means that it is likely *not* to have an efficient algorithm.

Asymptotic Analysis

Hierarchy of Function Classes

- Constant, O(1), functions don't grow at all.
- Logarithmic, O(log n), functions are slower growing than linear functions.
- Liner, O(n), functions are slower growing than O(n log n) functions.
- O(n log n) functions are slower growing than quadratic functions.
- Polynomial functions, i.e., $O(n^k)$ functions where k is constant.
- Exponential functions, i.e., $O(a^n)$ functions where a>1.

Big O times



Example Execution Times

Clock rate:	1,000,000,000
seconds/day	86400
seconds/year	31536000

size	log n	n	n log n	n^2	n^3	2^n
10	3 ns	0.00001 ms	0.00003 ms	0.0001 ms	0.0010 ms	0.00102 ms
20	4 ns	0.00002 ms	0.00009 ms	0.0004 ms	0.0080 ms	1.04858 ms
30	5 ns	0.00003 ms	0.00015 ms	0.0009 ms	0.0270 ms	1.0737 s
50	6 ns	0.00005 ms	0.00028 ms	0.0025 ms	0.1250 ms	13.0312 days
100	7 ns	0.00010 ms	0.00066 ms	0.0100 ms	1.0000 ms	4.0E+13 years
1000	10 ns	0.00100 ms	0.00997 ms	1.0000 ms	1000.0000 ms	3.4E+284 years
10000	13 ns	0.01000 ms	0.13288 ms	0.1000 s	1000.0000 s	#NUM!
100000	17 ns	0.10000 ms	1.66096 ms	10.0000 s	11.5741 days	#NUM!
1000000	20 ns	1.00000 ms	19.93157 ms	1000.0000 s	31.7098 years	#NUM!

Some General Observation

- O(1) denotes “constant time” – anything not dependent on the input size.
- A polynomial is always big-O of its leading term.
- For a O(f) operation followed by an O(g) operation, you can ignore the smaller one. E.g., $O(n^2 + n)$ is $O(n^2)$.
- If a O(f) operation is repeated O(g) times, the total time is O(f • g). E.g., if an $O(n^2)$ operation is performed O(n log n) times, the whole thing is $O(n^3 \log n)$.
- If the problem size n is decreased by a *constant factor* at each step, the number of steps is O(log n).