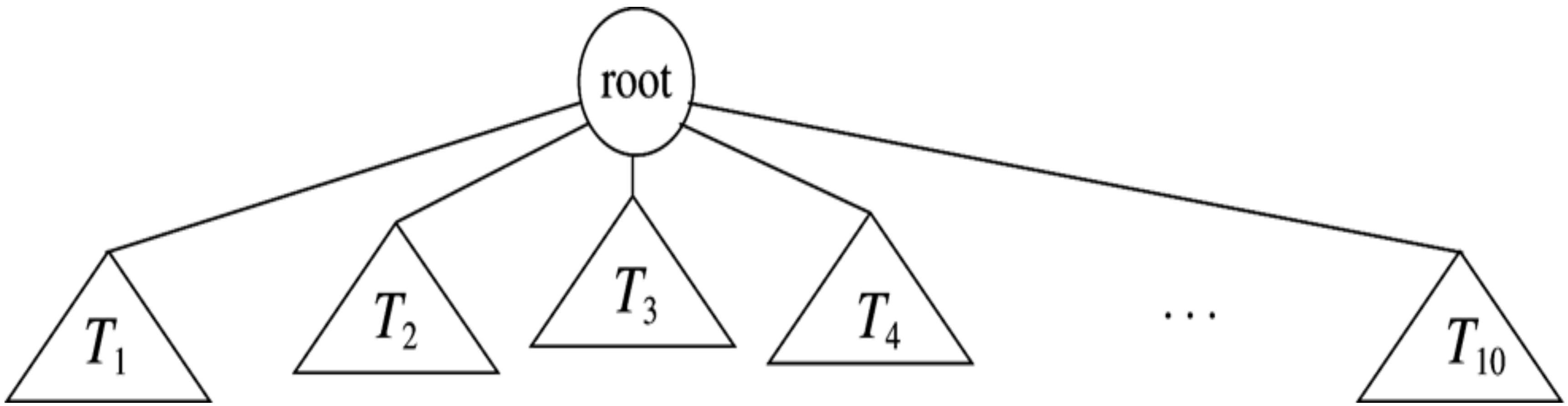
CMSC 206

Introduction to Trees

Tree ADT

- Tree definition
  - A tree is a set of nodes which may be empty
  - If not empty, then there is a distinguished node ***r***, called *root* and zero or more non-empty subtrees T_1, T_2, … T_k, each of whose roots are connected by a directed edge from r.
- This recursive definition leads to recursive tree algorithms and tree properties being proved by induction.
- Every node in a tree is the root of a subtree.

A Generic Tree



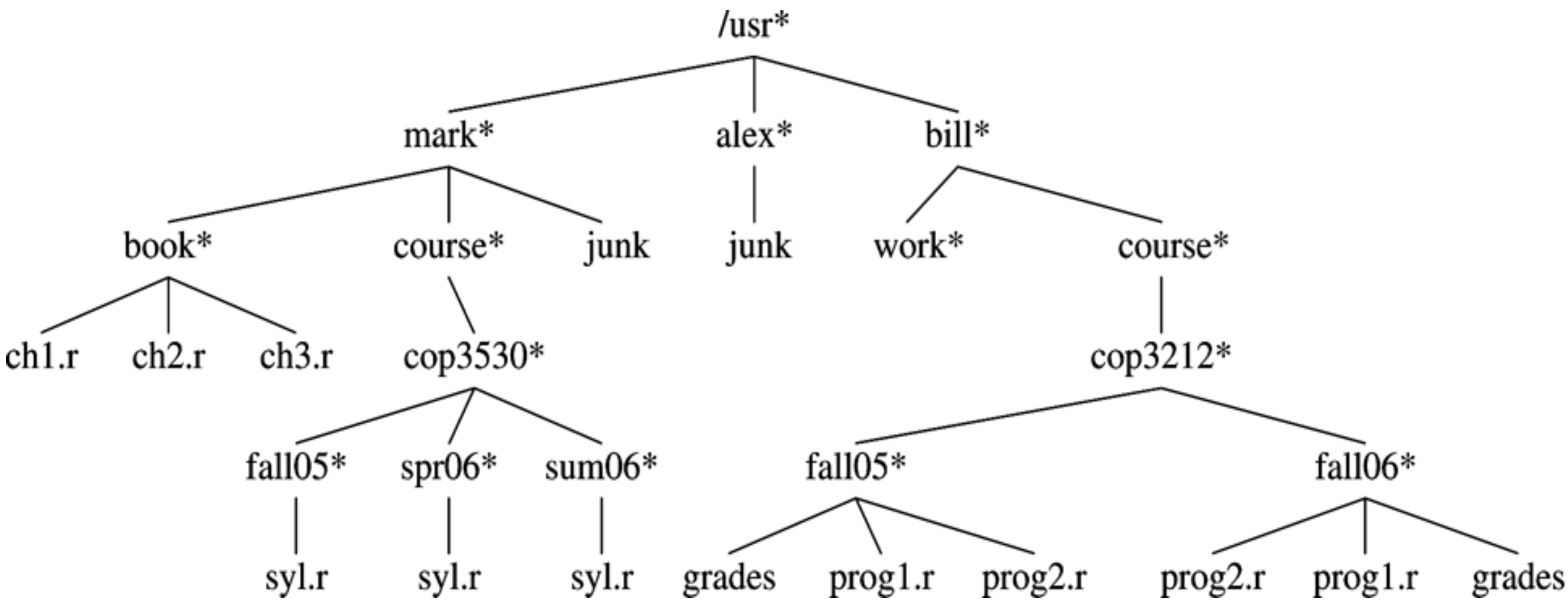
Tree Terminology

- *Root* of a subtree is a child of **r**. **r** is the *parent*.
- All children of a given node are called *siblings*.
- A *leaf* (or external node) has no children.
- An *internal node* is a node with one or more children
- A *path* from node V_1 to node V_k is a sequence of nodes s.t. V_i is the parent of V_{i+1} for $1 \leq i \leq k$.
  - If there is a path from V_1 to V_2, then V_1 is an *ancestor* of V_2 and V_2 is a *descendent* of V_1.

More Tree Terminology

- The *length* of this path is the number of edges.
  - The length of the path is one less than the number of nodes on the path (k – 1 in this example)
- The *depth* (also called *level)* of any node in a tree is the length of the path from root to the node.
- The *height* of a tree is the length of the path from the root to the deepest node in the tree.
  - A tree with only one node (the root) has height 0.

A Unix directory tree



Tree Storage

- A tree node contains:
  - Data Element
  - Links to other nodes
- Any tree can be represented with the “first-child, next-sibling” implementation.

```
class TreeNode
{
    AnyType    element;
    TreeNode firstChild;
    TreeNode nextSibling;
}
```

Printing a Child/Sibling Tree

```
// depth equals the number of tabs to indent name
private void listAll( int depth )
{
         printName( depth ); // Print the name of the object
         if( isDirectory( ) )
               for each file c in this directory
               (i.e. for each child)
                      c.listAll( depth + 1 );
}
public void listAll( )
{
        listAll( 0 );
}
```

- What is the output when listAll( ) is used for the Unix directory tree?

K-ary Tree

- If we know the maximum number of children each node will have, K, we can use an array of children references in each node.

```
class KTreeNode
{
    AnyType element;
    KTreeNode children[ K ];
}
```

Pseudocode for Printing a K-ary Tree

```
// depth equals the number of tabs to indent name
private void listAll( int depth )
{
    printElement( depth ); // Print the object
     if( children != null )
        for each child c in children array
            c.listAll( depth + 1 );
}

public void listAll( )
{
    listAll( 0 );
}
```

Binary Trees

- A special case of K-ary tree is a tree whose nodes have exactly two child references -- binary trees.

- A *binary tree* is a rooted tree in which no node can have more than two children AND the children are distinguished as *left* and *right*.

The Binary Node Class

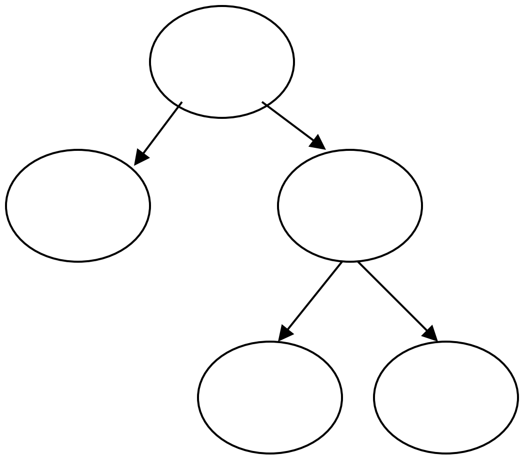
```
private class BinaryNode<AnyType>
{
    // Constructors
    BinaryNode( AnyType theElement )
    {
        this( theElement, null, null );
    }

    BinaryNode( AnyType theElement,
          BinaryNode<AnyType> lt, BinaryNode<AnyType> rt )
    {
        element  = theElement; left = lt; right = rt;
    }

    AnyType element;             // The data in the node
    BinaryNode<AnyType> left;    // Left child reference
    BinaryNode<AnyType> right;   // Right child reference
}
```

Full Binary Tree

A full binary tree is a binary tree in which every node is a leaf or has exactly two children.



FBT Theorem

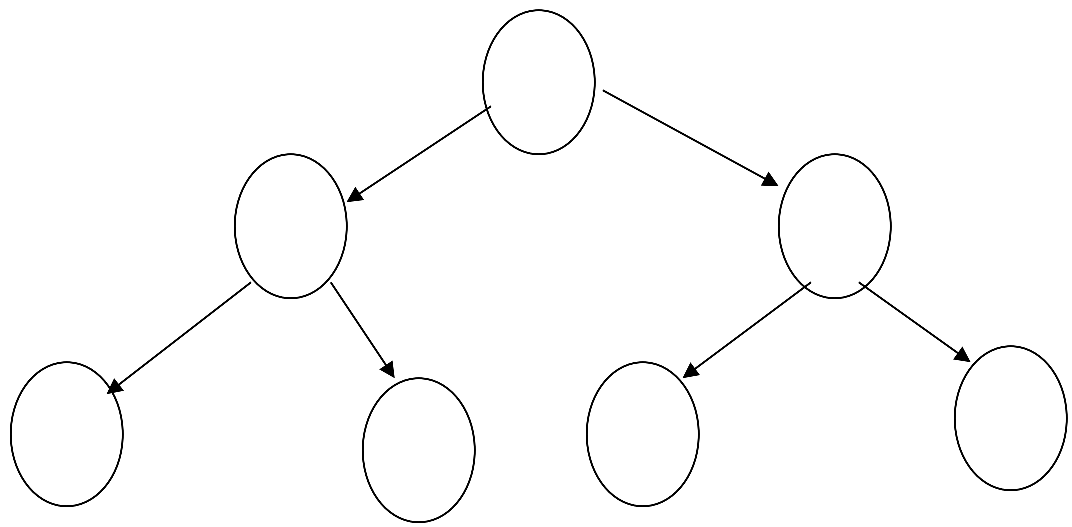
- **Theorem: A FBT with n internal nodes has n + 1 leaves (external nodes)**.
- Proof by strong induction on the number of internal nodes, n:
- Base case:
  - Binary Tree of one node (the root) has:
    - zero internal nodes
    - one external node (the root)
- Inductive Assumption:
  - Assume all FBTs with n internal nodes have n + 1 external nodes.

FBT Proof (cont'd)

- Inductive Step - prove true for a tree with n + 1 internal nodes (i.e. a tree with n + 1 internal nodes has (n + 1) + 1 = n + 2 leaves)
  - Let T be a FBT of n internal nodes.
  - Therefore T has n + 1 leaf nodes. (Inductive Assumption)
  - Enlarge T so it has n+1 internal nodes by adding two nodes to some leaf. These new nodes are therefore leaf nodes.
  - Number of leaf nodes increases by 2, but the former leaf becomes internal.
  - So,
    - # internal nodes becomes n + 1,
    - # leaves becomes (n + 1) + 2 - 1 = n + 2

Perfect Binary Tree

- A *Perfect Binary Tree* is a Full Binary Tree in which all leaves have the same depth.



PBT Theorem

- **Theorem: The number of nodes in a PBT is $2^{h+1}-1$, where h is height**.
- Proof by strong induction on h, the height of the PBT:
  - Notice that the number of nodes at each level is 2^l. (Proof of this is a simple induction - left to student as exercise). Recall that the height of the root is 0.
  - Base Case:
    The tree has one node; then h = 0 and n = 1
    and $2^{(h+1)} - 1 = 2^{(0+1)} - 1 = 2^1 - 1 = 2 - 1 = 1 = n$.
  - Inductive Assumption:
    Assume true for all PBTs with height $h \leq H$.

Proof of PBT Theorem(cont)

- Prove true for PBT with height H+1:
  - Consider a PBT with height H + 1. It consists of a root and two subtrees of height <= H. Since the theorem is true for the subtrees (by the inductive assumption since they have height ≤ H) the PBT with height H+1 has
  - $(2^{(H+1)} - 1)$ nodes for the left subtree

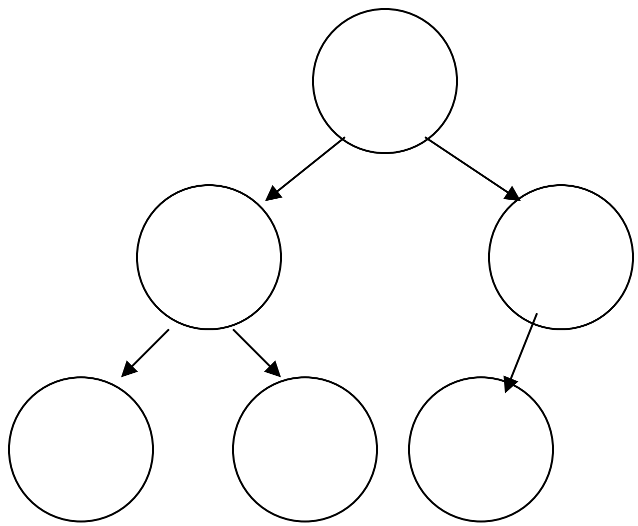
    $+ (2^{(H+1)} - 1)$ nodes for the right subtree

    $+ 1$ node for the root
  - Thus, $n = 2 * (2^{(H+1)} – 1) + 1$

    $= 2^{((H+1)+1)} - 2 + 1 = 2^{((H+1)+1)} - 1$

Complete Binary Tree

A Complete Binary Tree is a binary tree in which every level is completed filled, except possibly the bottom level which is filled from left to right.



Tree Traversals

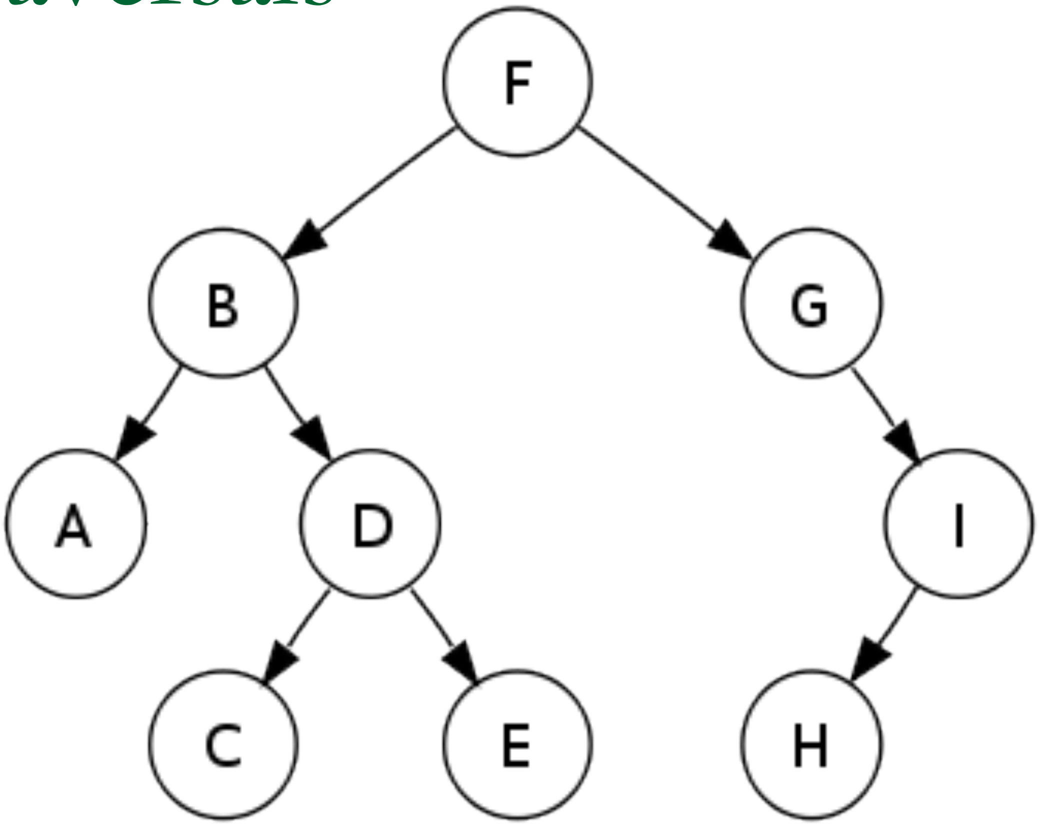
Depth-First Traversals

- Preorder – root, left subtree, right subtree
- Inorder – left subtree, root, right subtree
- Postorder – left subtree, right subtree, root

Breadth-First Traversal

- Level-order – each level is printed in turn

Tree Traversals



Depth-first

Preorder: F, B, A, D, C, E, G, I, H (root, left, right)

Inorder: A, B, C, D, E, F, G, H, I (left, root, right) ← Notice the sorting!

Postorder: A, C, E, D, B, H, I, G, F (left, right, root)

Breadth-first

Level-order: F, B, G, A, D, I, C, E, H

Constructing Trees

- Is it possible to reconstruct a Binary Tree from just one of its pre-order, inorder, or post-order sequences?

Constructing Trees (cont)

- Given two sequences (say pre-order and inorder) is the tree unique?

Finding an element in a Binary Tree?

- Return a reference to node containing x, return null if x is not found

```
public BinaryNode<AnyType> find(AnyType x)
{
    return find(root, x);
}
private BinaryNode<AnyType> find( BinaryNode<AnyType> node, AnyType x)
{
   BinaryNode<AnyType> t = null;           // in case we don't find it
   if ( node.element.equals(x) )           // found it here??
        return node;

   // not here, look in the left subtree
   if(node.left != null)
        t = find(node.left,x);

   // if not in the left subtree, look in the right subtree
   if ( t == null && node.right != null)
        t = find(node.right,x);

   // return reference, null if not found
   return t;
}
```

Binary Trees and Recursion

- A Binary Tree can have many properties
  - Number of leaves
  - Number of interior nodes
  - Is it a full binary tree?
  - Is it a perfect binary tree?
  - Height of the tree
- Each of these properties can be determined using a recursive function.

Recursive Binary Tree Function

```
return-type function (BinaryNode<AnyType> t)
{
     // base case – usually empty tree
    if (t == null) return xxxx;

    // determine if the node referred to by t has the property

    // traverse down the tree by recursively "asking" left/right
    // children if their subtree has the property

    return theResult;
}
```

Is this a full binary tree?

```
boolean  isFBT (BinaryNode<AnyType> t)
{
   // base case – an empty tee is a FBT
   if (t == null) return true;

   // determine if this node is "full"
   // if just one child, return – the tree is not full
   if ((t.left == null && t.right != null)
   ||  (t.right == null && t.left != null))
        return false;

   // if this node is full, "ask" its subtrees if they are full
   // if both are FBTs, then the entire tree is an FBT
   // if either of the subtrees is not FBT, then the tree is not
   return isFBT( t.right ) && isFBT( t.left );

}
```

Other Recursive Binary Tree Functions

- Count number of interior nodes

```
int countInteriorNodes( BinaryNode<AnyType> t);
```

- Determine the height of a binary tree. By convention (and for ease of coding) the height of an empty tree is -1

```
int height( BinaryNode<AnyType> t);
```

- Many others

Other Binary Tree Operations

- How do we insert a new element into a binary tree?
- How do we remove an element from a binary tree?