Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - $\text{A\sp{D}, 5\sp{H}, 7\sp{C}, 10\sp{S}, K\sp{S}}$
- Is that the same hand as:
  - $\text{K\sp{S}, 10\sp{S}, 7\sp{C}, 5\sp{H}, A\sp{D}}$
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

- An $r$-permutation is an ordered arrangement of $r$ elements of the set
  - $\text{A\sp{D}, 5\sp{H}, 7\sp{C}, 10\sp{S}, K\sp{S}}$ is a 5-permutation of the set of cards
- The notation for the number of $r$-permutations:
  - $P(n,r) = \frac{n!}{(n-r)!}$

Combinations

- What if order doesn’t matter?
  - In poker, the following two hands are equivalent:
  - $\text{A\sp{D}, 5\sp{H}, 7\sp{C}, 10\sp{S}, K\sp{S}}$
  - $\text{K\sp{S}, 10\sp{S}, 7\sp{C}, 5\sp{H}, A\sp{D}}$
- The number of $r$-combinations of a set with $n$ elements, where $n$ is non-negative and $0 \leq r \leq n$ is:
  - $C(n,r) = \frac{n!}{r!(n-r)!}$

Combinations example

- How many different poker hands are there (5 cards)?
  - $C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} \cdot \frac{52\times51\times50\times49\times48}{5\times4\times3\times2\times1} = 2,598,960$
- How many different (initial) blackjack hands are there?
  - $C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} \cdot \frac{52\times51}{2\times1} = 1,326$

Combination formula proof

- Let $C(52,5)$ be the number of ways to generate unordered poker hands
- The number of ordered poker hands is $P(52,5) = 311,875,200$
- The number of ways to order a single poker hand is $P(5,5) = 5! = 120$
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, $C(52,5) = P(52,5)/P(5,5)$
Combination formula proof

• Let $C(n,r)$ be the number of ways to generate unordered combinations
• The number of ordered combinations (i.e. $r$-permutations) is $P(n,r)$
• The number of ways to order a single one of those $r$-permutations $P(r,r)$
• The total number of unordered combinations is the total number of ordered combinations (i.e. $r$-permutations) divided by the number of ways to order each combination
• Thus, $C(n,r) = \frac{P(n,r)}{P(r,r)}$

Combination Formula

$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$

Bit Strings

• How many bit strings of length 10 contain:
  • Exactly four 1's?
    – Find the positions of the four 1's
    – Does the order of these positions matter?
    – Thus, the answer is $C(10,4) = 210$
  • At most four 1's?
    – There can be 0, 1, 2, 3, or 4 occurrences of 1
    – $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)$
    – $1+10+45+120+210$
    – $= 386$

Corollary 1

• Let $n$ and $r$ be non-negative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$

• Proof:
  
  $C(n,r) = \frac{n!}{r!(n-r)!}$
  
  $C(n,n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!}$

Bit Strings

• How many bit strings of length 10 contain:
  • At least four 1's?
    – There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
    – $C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10)$
    – $= 210+252+210+120+45+10+1$
    – $= 848$
    – Alternative answer: subtract from $2^{10}$ the number of strings with 0, 1, 2, or 3 occurrences of 1
  • An equal number of 1's and 0's?
    – Thus, there must be five 0's and five 1's
    – Find the positions of the five 1's
    – Thus, the answer is $C(10,5) = 252$

Corollary example

• There are $C(52,5)$ ways to pick a 5-card poker hand
• There are $C(52,47)$ ways to pick a 47-card hand
• $P(52,5) = 2,598,960 = P(52,47)$

• When dealing 47 cards, you are picking 5 cards to not deal
  – As opposed to picking 5 card to deal
  – Again, the order the cards are dealt in does matter
Note
• An alternative (and more common) way to denote an $r$-combination:

$$C(n, r) = \binom{n}{r}$$

Choosing Teams
• Choosing team of 5 among 12
• Two members must work as a pair
  – # of teams that contain both: $C(10, 3) = 120$
  – # of teams that don’t: $C(10, 5) = 252$
  – addition rule
• Two members must be kept apart
  – # of teams that have either: $2xC(10, 4) = 420$
  – # of teams that don’t: $C(10, 5) = 252$

Choosing Teams
• We have 5 men and 7 women
• How many 5-person groups can be chosen that
  – consist of 3 men and 2 women?
    • $C(5, 3) \times C(7, 2) = 210$
  – have at least one man?
    • $C(12, 5) - C(7, 5) = 771$
  – at most one man?
    • $C(7, 5) + C(5, 1) \times C(7, 4) = 196$

$r$-Combinations with Repetitions
• How many 2-combinations can be selected from {1, 2, 3}, if repetitions are allowed?
  – {1, 1}, {1, 2}, {1, 3}, {2, 2}, {2, 3}, {3, 3}

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>selection</th>
<th>string</th>
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<tbody>
<tr>
<td>XX</td>
<td></td>
<td>3</td>
<td>(1, 1)</td>
<td>xx</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>(1, 2)</td>
<td>x</td>
</tr>
<tr>
<td>X</td>
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<td></td>
<td>(1, 3)</td>
<td>x</td>
</tr>
<tr>
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<tr>
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<td>(2, 3)</td>
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</tr>
<tr>
<td>XX</td>
<td>x</td>
<td></td>
<td>(3, 3)</td>
<td></td>
</tr>
</tbody>
</table>

$r$-Combinations with Repetitions
• The number of $r$-Combinations with repetition allowed that can be selected from a set of $n$ elements is: $C(r+n-1, r)$
Soda Distribution
• Select 15 cans of soft drinks from 5 types
  – How many different selections?
    • \( C(5+15-1, 15) = C(19, 15) = 3,876 \)
  – If Diet Coke is one of the types, how many selections include at least 6 cans Diet Coke?
    • choose the DCs first, then the rest
    • \( C(5+9-1, 9) = C(13, 9) = 715 \)
  – If the store only has 5 cans of DC, but at least 15 cans of all others, how many selections?

Ways to Count
• Choosing \( k \) elements from \( n \)

<table>
<thead>
<tr>
<th>Repetition allowed</th>
<th>order matters</th>
<th>Repetition allowed</th>
<th>order doesn't matter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^k )</td>
<td></td>
<td>( C(k+n-1, k) )</td>
<td></td>
</tr>
<tr>
<td>No repetition</td>
<td></td>
<td>( P(n, k) )</td>
<td>( C(n, k) )</td>
</tr>
</tbody>
</table>

Circular seatings
• How many ways are there to seat 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?
  – First, place the first person in the north-most chair
    – Only one possibility
  – Then place the other 5 people
    – There are \( P(5,5) = 5! = 120 \) ways to do that
  – By the product rule, we get \( 1 \times 120 = 120 \)
  – Alternative means to answer this:
    • There are \( P(6,6) = 720 \) ways to seat the 6 people around the table
    • For each seating, there are 6 "rotations" of the seating
    • Thus, the final answer is \( 720/6 = 120 \)

Horse races
• How many ways are there for 4 horses to finish if ties are allowed?
  – Note that order does matter!
  • Solution by cases
    – No ties
      • The number of permutations is \( P(4,4) = 4! = 24 \)
    – Two horses tie
      • There are \( C(4,2) = 6 \) ways to choose the two horses that tie
      • There are \( P(3,3) = 6 \) ways for the "groups" to finish
      • A "group" is either a single horse or the two tying horses
      • By the product rule, there are \( 6 \times 6 = 36 \) possibilities for this case
    – Two groups of two horses tie
      • There are \( C(4,2) = 6 \) ways to choose the two winning horses
      • The other two horses tie for second place
      • By the product rule, there are \( 6 \times 6 = 36 \) possibilities for this case
    – Three horses tie with each other
      • There are \( C(4,3) = 4 \) ways to choose the three horses that tie
      • There are \( P(2,2) = 2 \) ways for the "groups" to finish
      • By the product rule, there are \( 4 \times 2 = 8 \) possibilities for this case
    – All four horses tie
      • There is only one combination for this
      • By the sum rule, the total is \( 24+36+6+8+1 = 75 \)

Counting Triples
• How many \((i, j, k)\) such that \( 1 \leq i \leq j \leq k \leq n\)?
  • If \( n=5 \), represent \((3, 3, 4)\) as \(|x|x||x|x|
  • If \( n=7 \), represent \((2, 4, 5)\) as \(|x|x||x||x|
  • How many \(|x|'s\)?
  • How many \(x\)'s?
  • \( C(3+n-1, 3) = (n+2)!/3!(n-1)! \)
  = \((n+2)(n+1)n/6 \)

Nested for loop
\[
\begin{align*}
\text{for} \ (k:=1\ \text{to} \ n) \\
\text{for} \ (j:=1\ \text{to} \ k) \\
\text{for} \ (i:=1\ \text{to} \ j) \\
\{\text{body} \} \\
\text{next} \ i \\
\text{next} \ j \\
\text{next} \ k
\end{align*}
\]
• How many times will the innermost loop body be executed?
  • For each iteration, there is a different combination of the indices \((i, j, k)\), \(1 \leq i \leq j \leq k \leq n\)