Conditional Probability

CS231
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Boy or Girl?

• A couple has two children, one of them is a girl. What is the probability that the other one is also a girl? Assuming 50/50 chances of conceiving boys and girls.

Example

• Two cards are drawn from a well-shuffled deck. What is the probability that:
  – both are kings?
  – second draw is a king?
• A = 1st draw is king, B = 2nd draw is king
  P(A) = 4/52, P(A^c) = 48/52
  P(B|A) = 3/51, P(B|A^c) = 4/51
  P(A∩B) = 4/52 × 3/51 = 12/2652

Example

• If the experiment of drawing a pair is repeated over time, what would be the expected value of the number of kings?
  • 2 kings: P(A∩B) = 4/52 × 3/51 = 12/2652
  • 1 king: P(A∩B) + P(A∩B^c) = 48/52 × 4/51 + 4/52 × 48/51 = 384/2652
  • Expected value of # of kings: 2 × 12/2652 + 1 × 384/2652 ≈ 0.154

Example

• 5% of manufactured components are defective in general.
  The method for screening out defective items is not totally reliable. The test rejects good parts as defective in 1% of the cases and accepts defective parts as good ones in 10% of the cases.
  Given that the test indicates that an item is good, what is the probability that this item is, in fact, defective?
Definitions

- \( T \) = A component tested good
- \( D \) = A component is defective
- \( T^c \) = A component tested defective
- \( G \) = A component is good (\( G = D^c \))
- Want to solve: \( P(D|T) \)

\[
P(D|T) = 0.05, P(G) = 0.95
\]

Medical Screening

- 1\% of population suffer from a certain disease.
- The method for screening is not totally reliable. The test reports false positive in 5\% of the cases and false negative in 10\% of the cases.
- Given that a person has a negative test result, what is the probability that this person is, in fact, sick?
- Given that a person has a positive test result, what is the probability that this person is, in fact, sick?

\[
P(S|T) = 0.01, P(H) = 0.99
\]

\[
P(T^c|H) = 0.05 \text{ (false positive)} \quad P(T|H) = 0.95
\]

\[
P(T|H) = P(T^c|H) \times P(H) = 0.95 \times 0.99 = 0.9405
\]

\[
P(T|S) = 0.1 \text{ (false negative)}
\]

\[
P(T\cap S) = P(T|S) \times P(S) = 0.1 \times 0.01 = 0.001
\]

\[
T = (T \cap H) \cup (T \cap S)
\]

\[
P(T) = 0.9405 + 0.001 = 0.9415
\]

\[
P(S|T) = P(T\cap S)/P(T) = 0.001/0.9415 = 0.001
\]

Bayes’ Theorem

- If \( E_1, E_2, \ldots, E_n \) are mutually exclusive and exhaustive events in a sample space, the total probability of any event \( F \) is:

\[
P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)
\]

- For any event \( E \) and \( F \) with \( P(F) \neq 0 \):

\[
P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F \cap E)P(E)}
\]
P(S|T) and P(S|T^c) with Bayes’

- P(S) = 0.01, P(H) = P(S^c) = 0.99
- P(T|H) = P(T|S^c) = 0.05, P(T|H) = P(T|S) = 0.95
- P(T|S) = 0.1, P(T|S^c) = 0.9

\[
P(S|T) = \frac{P(T|S)P(S)}{P(T|S)P(S) + P(T|S^c)P(S^c)} = \frac{0.1 \times 0.01}{0.1 \times 0.01 + 0.9 \times 0.99} \approx 0.001
\]

\[
P(S|T^c) = \frac{P(T^c|S)P(S)}{P(T^c|S)P(S) + P(T^c|S^c)P(S^c)} = \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.009 + 0.0495} \approx 0.1538
\]

Monty Hall Revisited

- A = prize is behind door A, B = prize is behind door B, C = prize is behind door C
- M_A = Monty opens door A, etc
- You choose door A and Monty opens a door revealing no prize

\[
P(M_0|A) = 1/2, P(M_0|B) = 0, P(M_0|C) = 1
\]

\[
P(M_0) = 1/2, P(A) = P(B) = P(C) = 1/3
\]

\[
P(A|M_0) = P(M_0|A)P(A)P(M_0) = 1/2 \times 1/3 \times 1/2 = 1/3
\]

\[
P(C|M_0) = P(M_0|C)P(C)P(M_0) = 1 \times 1/3 \times 1/2 = 2/3
\]

- Exact same analysis holds for M_C

Bayes’ Theorem

- If E_1, E_2, ..., E_n are mutually exclusive and exhaustive events in a sample space, given any F with P(F) ≠ 0,

\[
P(E_i | F) = \frac{P(F | E_i)P(E_i)}{\sum_{i=1}^{n} P(F | E_i)P(E_i)} = \frac{P(F | E_i)P(E_i)}{P(F)}
\]

- P(θ) is also known as prior

- P(θ|X) is the posterior probability after observing X and obtaining P(X|θ)

Bayes’ Ratio

- When there are three events, A, B and C and the comparative posterior probabilities are of interest, consider the ratio:

\[
\frac{P(A | C)}{P(B | C)} = \frac{P(C | A)}{P(C | B)} \times \frac{P(A)}{P(B)}
\]

Example

- Two bags, one contains 70 red and 30 blue balls, and the other 30 red and 70 blue balls.
- Choose one bag randomly and draw with replacement.
- 8 red and 4 blue balls are drawn in 12 tries.
- What is the probability that it was the predominantly red bag that was chosen?

Solution

- A = selecting the 1st bag, B = selecting the 2nd bag, C = getting the draws we did

\[
P(C|A) = (7/10)^8 \times (3/10)^4 \times C(12, 8)
\]

\[
P(C|B) = (7/10)^4 \times (3/10)^8 \times C(12, 8)
\]

\[
P(A) = P(B) = 0.5
\]

\[
P(C|A) / P(C|B) = (7/10)^4 \times (3/10)^4 = (7/3)^4
\]

\[
P(A|C) / P(B|C) = (7/3)^4
\]

\[
P(A|C) + P(B|C) = 1 \quad \Rightarrow P(A|C) = (7/3)^4 / ((7/3)^4 + 1)
\]
Dramatic Taxicab

• A cab was involved in a hit-and-run at night.
• Two cab companies operate in the city, with green and blue cabs, respectively.
• 85% of the cabs are green.
• A witness identified the cab as blue.
• The witness correctly identified the two colors 80% of the time under night-time testing.
• What is the probability that the witness was right?

Independent Events

• Two events are independent when the occurrence of one does not affect the probability of the other.
  – tossing coins
  – rolling dice
• Events A and B are independent iff:
  \[ P(A \cap B) = P(A) \times P(B) \]

P(A\cap B^c)

• If A and B are independent events, so are A and B^c.
• From set theory:
  – (A\cap B) U (A\cap B^c) = A
  – (A\cap B) \cap (A\cap B^c) = \emptyset
• \[ P((A\cap B)U(A\cap B^c)) = P(A\cap B)+P(A\cap B^c) = P(A) \]
• \[ P(A\cap B^c) = P(A) - P(A\cap B) = P(A) - P(A)P(B) \]
  \[ = P(A)(1 - P(B)) = P(A)P(B^c) \]

Loaded Coin

• A coin is loaded so that the probability of heads is 0.6. After 10 tosses, what is the probability of obtaining 8 heads?
• Consider HHHHHHHHHTT
  \[ P(HHHHHHHHHTT) = 0.6^8 \times 0.4^2 \]
• How many ways can you get 8 heads with 10 tosses? – C(10, 8)
  \[ P(8 \text{ heads}) = C(10, 8) \times 0.6^8 \times 0.4^2 \approx 0.12 \]