The Binomial Theorem

CS231
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Pascal’s Formula

• One of the most famous and useful in Combinatorics
• \( C(n+1, r) = C(n, r-1) + C(n, r) \)
• Recall another important combinatorial result:
  • \( C(n, r) = C(n, n-r) \)

Combinatorial Proof

• A combinatorial proof is a proof that uses counting arguments to prove a theorem
  – Rather than some other method such as algebraic techniques
• Essentially, show that both sides of the proof manage to count the same objects
• In other words, a bijection between the two sets

Combinatorial Proof

• \( C(n+1, r) \):  
  – # of ways to choose \( r \) elements from \( n+1 \)
• Remove an arbitrary element from \( n+1 \), call it \( a \).
• Now form all possible subsets of size \( r \).
  – These are all the subsets of size \( r \) you can have without \( a \).
  – \( C(n, r) \)
• Now we need to account for subsets of size \( r \) with \( a \)

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

\[
\frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-k)!} + \frac{n!}{k!(n-k)!}
\]

Algebraic Proof

\[
\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} = \frac{n!}{(k-1)!(n-k)!} + \frac{n!}{k!(n-k)!}
\]

Substitutions:

\[
(n-k+1) - (n-k+1)(n-k)!
\]

\[
(n+1) - (n+1)!\]

\[
k - k(k-1)!
\]
Pascal’s triangle

Binomial Coefficients

- A quick expansion of \((x+y)^n\)
- Why it’s really important:
  - It provides a good context to present proofs
  - Especially combinatorial proofs

Polynomial Expansion

- Consider \((x+y)^3\):
  \((x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\)
- Rephrase it as:
  \[(x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3\]
- When choosing \(x\) twice and \(y\) once, there are \(\binom{3}{2} = 3\) ways to choose where the \(x\) comes from
- When choosing \(x\) once and \(y\) twice, there are \(\binom{3}{1} = 3\) ways to choose where the \(y\) comes from

Polynomial expansion

- Consider \((x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\)
- To obtain the \(x^5\) term
  - Each time you multiply by \((x+y)\), you select the \(x\)
  - Thus, of the 5 choices, you choose \(x\) 5 times or \(y\) 0 times
  - \(\binom{5}{0} = \binom{5}{5} = 1\)
- To obtain the \(x^4y\) term
  - Four of the times you multiply by \((x+y)\), you select the \(x\)
  - The other time you select the \(y\)
  - Thus, of the 5 choices, you choose \(x\) 4 times or \(y\) 1 time
  - \(\binom{5}{1} = 5\)
- To obtain the \(x^3y^2\) term
  - \(\binom{5}{3} = \binom{5}{2} = 10\)

Polynomial Expansion: The Binomial Theorem

- For \((x+y)^n\)
  \[\binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \ldots + \binom{n}{n} x^0 y^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\]
Sample question

• Find the coefficient of $x^5y^8$ in $(x+y)^{13}$

• Answer: $(\binom{13}{5} \binom{13}{8} = 1287$

Examples

• What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

  \[
  \binom{25}{12} \binom{13}{12} = \frac{25!}{12!13!} = 5,200,300
  \]

• What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?

  \[
  (2x+(-3y))^2 = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j
  \]

  The coefficient occurs when $j=13:

  \[
  \binom{25}{13} 2^2 (-3)^{13} = \frac{25!}{13!12!} 2^2 (-3)^{13} = -33,959,763,545,702,400
  \]

Pascal’s Triangle

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\binom{n}{k}$</th>
<th>$\sum \binom{n}{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1 = $2^n$</td>
</tr>
<tr>
<td>1</td>
<td>1, 0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1, 1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1, 3, 3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>1, 4, 6, 4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>1, 5, 10, 10, 5</td>
<td>32</td>
</tr>
<tr>
<td>6</td>
<td>1, 6, 15, 20, 15, 6</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>1, 7, 21, 35, 35, 21, 7</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>1, 8, 28, 56, 70, 70, 56, 28, 8, 1</td>
<td>256</td>
</tr>
</tbody>
</table>

Corollary 1 and Algebraic Proof

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad n \geq 0
\]

Algebraic proof

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

Combinatorial Proof

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad n \geq 0
\]

A set with $n$ elements has $2^n$ subsets

– By definition of and cardinality of power set

– Each subset has either 0 or 1 or 2 or … or $n$ elements

– There are \( \binom{n}{0} \) subsets with 0 elements,

– \( \binom{n}{1} \) subsets with 1 element, …

– and \( \binom{n}{n} \) subsets with $n$ elements

– Thus, the total number of subsets is \( \sum_{k=0}^{n} \binom{n}{k} \)
Corollary 2

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{k} = 0, n \geq 1 \]

- **Algebraic proof**
  
  \[ = ((-1)^i + 1)^n \]
  
  \[ = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \]
  
  \[ = \sum_{i=0}^{n} \binom{n}{i} (-1)^i y^i \]
  
  This implies that
  
  \[ \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \]

Corollary 3

- Let \( n \) be a non-negative integer. Then
  
  \[ \sum_{i=0}^{n} 2^i \binom{n}{k} = 3^i \]

- **Algebraic proof**
  
  \[ 3^i = (1 + 2)^i \]
  
  \[ = \sum_{i=0}^{n} \binom{n}{i} 2^{i-2} i \]
  
  \[ = \sum_{i=0}^{n} \binom{n}{i} 2^i \]

Vandermonde’s identity

- Let \( m, n, \) and \( r \) be non-negative integers with \( r \) not exceeding either \( m \) or \( n \). Then
  
  \[ \binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i} \binom{n}{r-i} \]

- Assume a congressional committee must consist of \( r \) people, and there are \( n \) Democrats and \( m \) Republicans.
  
  - How many ways are there to pick the committee?

More Combinatorial Proofs

- \( n^3 - n = 6C(n,2) + 6C(n,3) \)
- \( n^3 - n = (n+1)n(n-1) \)
  
  \[ = n(n-1)(n-2) + 3n(n-1) \]
- \( n^3 - n = P(n+1, 3) \)