Probability Trees and the Multiplication Rule

CS231
Dianna Xu

The multiplication rule

• Also called the product rule
• If there are \( n_1 \) ways to do task 1, and \( n_2 \) ways to do task 2
  – Then there are \( n_1 \times n_2 \) ways to do both tasks in sequence
  – We must make one choice AND a second choice

Product rule example

• Sample question
  – There are 18 MATH majors and 17 CS majors
  – How many ways are there to pick one math major and one CS major?

• Total is \( 17 \times 18 = 306 \)

Product rule example

• How many strings of 4 decimal digits…
  • Do not contain the same digit twice?
    – We want to chose a digit, then another that is not the same, then another...
    • First digit: 10 possibilities
    • Second digit: 9 possibilities (all but first digit)
    • Third digit: 8 possibilities
    • Fourth digit: 7 possibilities
    – Total = \( 10 \times 9 \times 8 \times 7 = 5040 \)
  • End with an even digit?
    – First three digits have 10 possibilities
    – Last digit has 5 possibilities
    – Total = \( 10 \times 10 \times 10 \times 5 = 5000 \)

When the product rule is difficult to apply

• President, treasurer and secretary are to be chosen among A, B, C, D. A can not be president and either C or D must be secretary.

• Naive application of the product rule:
  – President: 3
  – Treasurer: 3
  – Secretary: 2
  – Total = 18

Tree diagrams

• We can use tree diagrams to enumerate the possible choices

• Once the tree is laid out, the result is the number of (valid) leaves
Only 8 choices

Tree diagrams example
- Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s

How many ways can the Eagles get to 5-6 in the next 3 games?

Permutations
- Given a set of n elements, its permutations can be counted this way:
  - Choose one element for first position: n
  - Choose next element for second position: n-1
  - ...
  - Total: nx(n-1)x...x1 = n!

r-permutation
- An r-permutation of a set of n elements is an ordered selection of r elements from the n elements.
  - A♠, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of r-permutations: P(n,r)
  - The poker hand is one of P(52,5) permutations

r-permutations
- Number of poker hands (5 cards):
  - $P(52,5) = \frac{52!}{(52-5)!} = 311,875,200$
- Number of (initial) blackjack hands (2 cards):
  - $P(52,2) = \frac{52!}{(52-2)!} = 2,652$

\[
P(n,r) = \frac{n!}{(n-r)!} = \prod_{i=r+1}^{n} i
\]
r-permutation Formula

• There are \( n \) ways to choose the first element
  – \( n-1 \) ways to choose the second
  – \( n-2 \) ways to choose the third
  – …
  – \( n-r+1 \) ways to choose the \( r \)th element

• By the product rule, that gives us:
  \[ P(n,r) = n(n-1)(n-2)\ldots(n-r+1) \]

r-permutations example

• How many ways are there for 3 students in this class to sit together?

• There are 50 students in the class
  – \( P(50,3) = 50 \times 49 \times 48 = 117,600 \)
  – Note that the positions they take do matter

Permutations vs. \( r \)-permutations

• \( r \)-permutations: Choosing an ordered 5 card hand is \( P(52,5) \)
  – When people say “permutations”, they almost always mean \( r \)-permutations
    • But the name can refer to both

• Permutations: Choosing an order for all 52 cards is \( P(52,52) = 52! \)
  – Thus, \( P(n,n) = n! \)

Sample question

• How many permutations of \{a, b, c, d, e, f, g\} end with a?
  – Note that the set has 7 elements
  – The last character must be a
    – The rest can be in any order

• Thus, we want a 6-permutation on the set \{b, c, d, e, f, g\}
  • \( P(6,6) = 6! = 720 \)

• Why is it not \( P(7,6) \)?