Solving Recurrence Relations

CS231
Dianna Xu

Explicit formula

• An explicit formula for a recurrence relation is called a solution.
• Given a sequence $a_0, a_1, a_2, \ldots, a_n$ defined by a recurrence relation, an explicit formula states $a_n$ in terms of $n$ only, without involving any previous terms.

$$a_n = \sum_{i=1}^{n} i \Rightarrow a_n = \frac{n(n+1)}{2}$$

Finding an explicit formula

• Iteration: start from the initial condition and calculate successive terms of the sequence until you see a pattern.
• Start guessing based on the pattern.
• Prove your guessed formula by induction.

Arithmetic sequence

• Each term is the sum of the previous term and a constant: $a_k = a_{k-1} + d$.
• Consider

$$a_0 = 1$$
$$a_k = a_{k-1} + 5$$

Iteration

• $a_0 = 1$
  $$a_1 = a_0 + 5$$
  $$a_2 = a_1 + 5 = a_0 + 5 + 5$$
  $$a_3 = a_2 + 5 = a_1 + 5 + 5 = a_0 + 5 + 5 + 5$$
  $$a_k = a_0 + 5 + \ldots + 5 + 5$$
  $$a_k = 1 + 5k$$

Arithmetic Sequence

• $a_k = a_{k-1} + d$
  $$a_0 = x$$
• $a_n = x + dn$
Geometric sequence

• Each term is the product of the previous term and a constant:

\[ a_0 = x \]
\[ a_k = ra_{k-1} \]

Explicit formula of a geometric sequence

• \( a_0 = x \)
• \( a_1 = ra_0 \)
• \( a_2 = ra_1 = r^2a_0 \)
• \( a_3 = ra_2 = r^3a_0 \)
• \( a_k = r^ka_0 \)
• \( a_k = r^kx \)

Growth of a Geometric Sequence

| \(10^7\) | Number of seconds in a year |
| \(10^9\) | Number of bytes of RAM in PC |
| \(10^{11}\) | Number of neurons in a human brain |
| \(10^{17}\) | Age of the universe in seconds |
| \(10^{31}\) | Number of seconds to process all possible positions of a checkers game, process rate of 1 move per nano second |
| \(10^{81}\) | Number of atoms in the universe |
| \(10^{111}\) | Number of seconds to process all possible positions of a chess game |

Tower of Hanoi Sequence

Recall that the ToH sequence satisfies the recurrence relation \( m_k = 2m_{k-1} + 1, k \geq 2 \)

\[ m_1 = 1 \]
\[ m_2 = 2m_1 + 1 = 2 + 1 \]
\[ m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1 = 2^2 + 2^1 + 2^0 \]
\[ m_4 = 2m_3 + 1 = 2(2^2 + 2^1 + 2^0) + 1 = 2^3 + 2^2 + 2^1 + 2^0 \]
\[ m_k = 2^{k-1} + ... + 2^1 + 2^0 = \sum_{i=0}^{k-1}2^i \]

Verify with Induction

• Base case: \( m_1 = 1 \)
• Inductive hypothesis: \( m_k = 2^k - 1 \)
• Prove for \( k+1 \):

\[ m_{k+1} = 2m_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1 \]
Example

- $a_n = a_{n-1} + 2n - 1, a_0 = 0$
- Iteration:
  - $a_1 = 0 + 2(1) - 1 = 1$
  - $a_2 = 1 + 2(2) - 1 = 4$
  - $a_3 = 4 + 2(3) - 1 = 9$
  - $a_4 = 9 + 2(4) - 1 = 16$
  - $a_5 = 16 + 2(5) - 1 = 25$
- Guess: $a_n = n^2$

Verify with Induction

- $a_n = a_{n-1} + 2n - 1, a_0 = 0$
- Base case: $a_0 = 0^2 = 0$
- Inductive Hypothesis: $a_k = k^2$
- Inductive Step:
  - $a_{k+1} = a_k + 2(k+1) - 1$
  - $a_{k+1} = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$ ■

Second-Order Linear Homogeneous with Constant Coefficients

$a_k = Aa_{k-1} + Ba_{k-2}, A, B \in R, B \neq 0$

- Second-order – two previous terms
- Linear – linear equation
- Homogeneous – no constant term
- Constant Coefficients – A and B do not depend on $k$

Which Sequence?

- Consider the sequence $1, t, t^2, t^3, ..., t^N$ ...
- $t \neq 0$
- $t^k = At^{k-1} + Bt^{k-2}$
- $t^k - At^{k-1} - Bt^{k-2} = 0$
- $t^2 - At - B = 0 \iff$ characteristic equation
- Solutions to a quadratic equation

Example

$a_k = 4a_{k-1} - 3a_{k-2}, k \geq 2$

- $t^2 - 4t + 3 = 0$
- $(t-3)(t-1) = 0$
- $t = 1$ or $t = 3$
- $t = 1: 1, 1, 1, 1, ..., 1 = 4x1 - 3x1$
- $t = 3: 1, 3, 9, 27, ..., 27 = 4x9 - 3x3$
- Consider 2, 4, 10, 28, ... 28 = 4x10 - 3x4
- What about 98, 94, 82, 46, ...

Lemma

- If $r_0, r_1, r_2, ...$ and $s_0, s_1, s_2, ...$ are sequences that satisfy the same second-order linear homogeneous recurrence relation with constant coefficients, for any constant $C$ and $D$,
  - $a_n = Cr_n + Ds_n, n \geq 0$
  - also satisfies the same recurrence relation.
Inductive Bases
• \( r_k = Ar_{k-1} + Br_{k-2} \) and \( s_k = As_{k-1} + Bs_{k-2}, \ k \geq 2 \)
• \( a_n = Cr_n + Ds_n \)
• \( a_n = C(Ar_{n-1} + Br_{n-2}) + D(As_{n-1} + Bs_{n-2}) \)
• \( a_n = A(Cr_{n-1} + Ds_{n-1}) + B(Cr_{n-2} + Ds_{n-2}) \)
• \( a_n = Aa_{n-1} + Ba_{n-2} \)

**Proof by Strong Induction**

• Bases:
  - \( a_0 = Cr^0 + Ds^0 = C + D \)
  - \( a_1 = Cr^1 + Ds^1 = Cr + Ds \)
• Inductive Hypothesis: \( a_k = Cr^k + Ds^k, \ k \geq 2 \)
• Inductive Step:
  - \( a_{k+1} = Aa_k + Ba_{k-1} = A(Cr^k + Ds^k) + B(Cr^{k-1} + Ds^{k-1}) \)
  - \( a_{k+1} = C(Ar^k + Br^{k-1}) + D(As^k + Bs^{k-1}) \)
  - \( a_{k+1} = Cr^{k+1} + Ds^{k+1} \)

**Distinct Root Theorem**

• If \( r \) and \( s \) are distinct roots to the characteristic equation \( t^2 - At - B = 0 \) of a recurrence relation \( a_k = Aa_{k-1} + Ba_{k-2}, \ k \geq 2 \), then the sequence is defined by the explicit formula:
  \( a_n = Cr^n + Ds^n \),
  where \( C \) and \( D \) are constants determined by \( a_0 \) and \( a_1 \), if given.
• \( a_0 = C + D, \ a_1 = Cr + Ds \)

**The Fibonacci Sequence**

• \( a_k = a_{k-1} + a_{k-2}, \ k \geq 2, \ a_0=1, \ a_1=1 \)
• Characteristic equation: \( t^2 - t - 1 = 0 \)
• Roots: \( a = 1, \ b = -1, \ c = -1 \)
  \( \Delta = (-1)^2 - 4 \times 1 \times (-1) = 5 \)
  \( x = (1 \pm \sqrt{5})/2 \)
  \( x_1 = (1+\sqrt{5})/2 \)
  \( x_2 = (1-\sqrt{5})/2 \)

**Fibonacci Sequence Explicit Formula**

\[
\begin{align*}
  a_n &= C \left(\frac{1+\sqrt{5}}{2}\right)^n + D \left(\frac{1-\sqrt{5}}{2}\right)^n \\
  a_0 &= 1 = C + D \\
  a_1 &= C \left(\frac{1+\sqrt{5}}{2}\right) + D \left(\frac{1-\sqrt{5}}{2}\right) = C = \frac{1}{\sqrt{5}} \\
  D &= -\frac{1}{\sqrt{5}} \\
  a_n &= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \times 2^n}
\end{align*}
\]

**Single Root Theorem**

• If \( r \) is a single real root to the characteristic equation \( t^2 - At - B = 0 \) of a recurrence relation \( a_k = Aa_{k-1} + Ba_{k-2}, \ k \geq 2 \), then the sequence is defined by the explicit formula:
  \( a_n = Cr^n + Dnr^n \),
  where \( C \) and \( D \) are constants determined by \( a_0 \) and \( a_1 \), if given.