Strong Mathematical Induction and the Well-ordering Principle

CS 231
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Strong induction

- Weak mathematical induction assumes P(k) is true, and uses that (and only that!) to show P(k+1) is true.
- Strong mathematical induction assumes P(1), P(2), ..., P(k) are all true, and uses that to show that P(k+1) is true.

\[ \text{[P(1) \land P(2) \land P(3) \land \ldots \land P(k)]} \rightarrow P(k+1) \]

Strong induction example 1

- Show that any number > 1 can be written as the product of one or more primes
- Base case: P(2)
  - 2 is the product of 2 (remember that 1 is not prime!)
- Inductive hypothesis: assume P(2), P(3), ..., P(k) are all true
- Inductive step: Show that P(k+1) is true

Strong induction vs. ordinary induction

- Determine which amounts of postage can be written with 5 and 6 cent stamps
  - Prove using both versions of induction
- Answer: any postage \( \geq 20 \)

Strong induction example 1

- Inductive step: Show that P(k+1) is true
- There are two cases:
  - \( k+1 \) is prime
    - It can then be written as the product of \( k+1 \)
  - \( k+1 \) is composite
    - It can be written as the product of two composites, a and b, where \( 2 \leq a \leq b < k+1 \)
    - By the inductive hypothesis, both P(a) and P(b) are true

Answer via mathematical induction

- Show base case: P(20):
  - \( 20 = 5 + 5 + 5 + 5 \)
- Inductive hypothesis: Assume P(k) is true
- Inductive step: Show that P(k+1) is true
  - If P(k) uses a 5 cent stamp, replace that stamp with a 6 cent stamp
  - If P(k) does not use a 5 cent stamp, it must use only 6 cent stamps
    - Since \( k > 18 \), there must be four 6 cent stamps
    - Replace these with five 5 cent stamps to obtain \( k+1 \)
Answer via strong induction

- Show base cases: P(20), P(21), P(22), P(23), and P(24)
  - 20 = 5 \times 5 + 5 + 5
  - 21 = 5 \times 5 + 5 + 6
  - 22 = 5 \times 5 + 6 + 6
  - 23 = 5 + 6 + 6
  - 24 = 6 + 6 + 6
- Inductive hypothesis: Assume P(20), P(21), ..., P(k) are all true
- Inductive step: Show that P(k+1) is true
  - Obtain P(k+1) by adding a 5 cent stamp to P(k+1-5)
  - P(k+1-5) = P(k-4) is true □

The Well-ordering Principle for Integers

- Let S be a set containing one or more integers all of which are greater than some fixed integer. Then S has a least element.
- Every non-empty set of positive integers contains a least element
- Equivalent to ordinary and strong mathematical inductions
  - i.e. if one is true, so are the other two

Archimedean property

- Let a, b be positive integers. ∃ positive integer n, such that na ≥ b.
- Assume there exists positive integers x and y such that ∀n, nx < y.
- Consider the set S = \{y – nx\}.
- By the well-ordering principle, S has a least element, say y-mx.
- Consider y-(m+1)x

Principle of mathematical induction

- Let P be a set of positive integers with the following properties:
  - 1 in P
  - k in P → k+1 in P
- Then P is the set of all positive integers

Proof with the well-ordering principle

- Let S be the set of all positive integers not in P.
- Assume that S is not empty.
- Then S has a least element, say a
  - a > 1 (1 in P)
  - a-1 is not in S (a is the least element of S)
  - a-1 in P → a in P
- Contradiction □

Chess and induction

Can the knight reach any square in a finite number of moves?

Show: the knight can reach any square (i, j) for which i+j=k where k > 1.

Base case: k = 2

Inductive hypothesis: the knight can reach any square (i, j) for which i+j=k where k > 1.

Inductive step: show the knight can reach any square (i, j) for which i+j=k+1 where k > 1.
Chess and induction

- Inductive step: show the knight can reach any square \((i, j)\) for which \(i+j = k+1\) where \(k \geq 1\).
  - Note that \(k+1 \geq 3\), and one of \(i\) or \(j\) is \(\geq 2\).
  - If \(i \geq 2\), the knight could have moved from \((i-2, j+1)\)
    - Since \(i+j = k+1\), \(i-2 + j+1 = k\), which is assumed true.
  - If \(j \geq 2\), the knight could have moved from \((i+1, j-2)\)
    - Since \(i+j = k+1\), \(i+1 + j-2 = k\), which is assumed true.

Polygon

- Triangulation
  - A triangulation of a polygon is a decomposition into triangles with maximal non-crossing diagonals.

Existence of a Diagonal

- Every polygon with \(n > 3\) vertices has a diagonal.

Theorem

- Every polygon admits a triangulation.
- Every triangulation of a polygon \(P\) with \(n\) vertices has \(n-2\) triangles and \(n-3\) diagonals.
- Proof by strong induction.