Mathematical Induction

CS 231
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How do you climb infinite stairs?

• Not a rhetorical question!
• First, you get to the base platform of the staircase
• Then repeat:
  – From your current position, move one step up

What is induction?

• A method of proof: \( \forall n, n \geq a, P(n) \)
• Three parts:
  – Base case(s): show it is true for one element
    • \( P(a) \) (get to the stair’s base platform)
  – Inductive hypothesis: assume it is true for any given element
    • Assume \( P(k), k \geq a \) (assume you are on a stair)
  – Show that it is true for the next highest element
    • \( P(k+1) \) (show you can move to the next stair)

Why does induction work?

• Establish that the truth of a proposition follows from smaller instances of the same proposition: \( P(k) \rightarrow P(k+1) \)
• Establish the truth of the smallest instance: \( P(a) \)
• In induction, the truth percolates up through the layers to prove the whole proposition

Induction example

• Show that the sum of the first \( n \) odd integers is \( n^2 \)
  – Example: If \( n = 5 \), \( 1+3+5+7+9 = 25 = 5^2 \)
  – Formally, show:
    \[ \forall n \; P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i-1 = n^2 \]
• Base case: Show that \( P(1) \) is true
  \[ P(1) = \sum_{i=1}^{1} 2i-1 = 1 \]

Induction example, continued

• Inductive hypothesis: assume true for \( k \)
  – Thus, we assume that \( P(k) \) is true, or that
    \[ \sum_{i=1}^{k} 2i-1 = k^2 \]
  – Note: we don’t yet know if this is true or not!
• Inductive step: show true for \( k+1 \)
  – We want to show that:
    \[ \sum_{i=1}^{k+1} 2i-1 = (k+1)^2 \]
Induction example, continued

• Recall the inductive hypothesis: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

• Proof of inductive step:

$$2(k+1) - 1 + \sum_{i=0}^{k} 2^i = 2(k+1) - 1 + 2^k - 1$$

$$= 2k + 1 + 2^k + 1$$

$$= (k+1)^2$$

Induction example

• Show that the sum of the first $n$ powers of 2 is $2^{n+1} - 1$, where $n$ starts at 0.
  – Example: If $n = 4$:
    - $1+2+2^2+2^3+2^4 = 1+2+4+8+16 = 31 = 2^5 - 1$
  – Formally, show: $\forall n P(n)$ where $P(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

• Base case: Show that $P(0)$ is true

$$P(0) = \sum_{i=0}^{0} 2^i = 1 = 2^1 - 1$$

Induction example, continued

• Inductive hypothesis: assume true for arbitrary $k$
  – Thus, we assume that $P(k)$ is true, or that
    $$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

• Inductive step: show true for $k+1$
  – Want to show that:
$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$

What did we show

• Base case: $P(0)$
  • If $P(0)$ is true, then $P(0+1)$ is true
    – i.e., $P(0) \Rightarrow P(0+1)$
  • We know it’s true for $P(0)$
  • Because of $P(k) \Rightarrow P(k+1)$, if it’s true for $P(0)$, then it’s true for $P(1)$
  • Because of $P(k) \Rightarrow P(k+1)$, if it’s true for $P(1)$, then it’s true for $P(2)$
  • Because of $P(k) \Rightarrow P(k+1)$, if it’s true for $P(2)$, then it’s true for $P(3)$
  • Because of $P(k) \Rightarrow P(k+1)$, if it’s true for $P(3)$, then it’s true for $P(4)$
  • And onwards to infinity
  • Thus, it is true for all possible values of $n$
    $$[P(0) \land \forall k (P(k) \Rightarrow P(k+1))] \Rightarrow \forall n P(n)$$

How to do inductive proofs

• Show the base case
• Establish the inductive hypothesis
• Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
• Prove the inductive step
Another induction example

• Show that \( n! < n^n, \forall n > 1 \)

• Base case: \( n = 2 \)
  \[
  2! < 2^2 \\
  2 < 4 
  \]

• Inductive hypothesis: assume \( k! < k^k \)

• Inductive step: show that \((k+1)! < (k+1)^{k+1}\)

\[
(k + 1)! = (k + 1)k! < (k + 1)k^k < (k + 1)(k + 1)^k = (k + 1)^{k+1}
\]

Another induction example

• Show that \( 6 | n^3 - n, \forall n \in \mathbb{Z}, n \geq 0 \)

• Base Case: \( P(0): 6 | 0^3 - 0 \)

• Inductive hypothesis: \( 6 | k^3 - k \)

• Inductive step:
  \[
  (k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1 \\
  = k^3 + 3k^2 + 3k - k + 3(k^2 + k) \\
  = k^3 - k + 3k(k + 1)
  \]

Applications of Induction

• Algebraic (in)equalities are not the only suitable applications of induction.

• How to apply the inductive step is less obvious in non-algebraic applications of induction
  – the manipulation needed to apply the hypothesis is not algebraic

Square Cutting

• Prove that given two or more squares, one can always cut them and reform them into a large square.

\[\Phi \quad \Rightarrow \quad \square\]

Trominoes

• Polyomino – generalization of domino

• Tromino –

• Prove that if any one square is removed from a \( 2^n \times 2^n \) checkerboard \( (n \geq 1, n \in \mathbb{Z}) \), the remaining squares can be completely covered by L-shaped trominoes.

Trominoes

• \( P(1): 2 \times 2 \) board:

• Assume a \( 2^k \times 2^k \) checkerboard can be covered except for any one square

• \( P(k+1): 2^{k+1} \times 2^{k+1} \) checkerboard
  – divide into 4 quadrants
  – each quadrant is of size \( 2^k \times 2^k \).
All Horses are the Same Color

- If there is only one horse, it’s of one color
- Assume within any set of \( k \) horses, there is only one color
- Consider \( k+1 \) horse, and divide into sets of \( \{1, 2, 3, \ldots, k\} \) and \( \{2, 3, 4, \ldots, k, k+1\} \).
  - Each is a set of \( k \) horses and can be of only one color.
  - Since there is overlap among the sets, there is only one color for all \( k+1 \) horses.

All Men are Bald

- A man with 0 (or 1) hair is clearly bald
- Assume a man with \( k \) hairs is bald
- One more hair on a bald head does not cure baldness, thus a man with \( k+1 \) hair is also bald.