Sequences

CS 231
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Definitions

• Sequence: an ordered list of elements
• A sequence is a function whose domain is a subset of \( \mathbb{Z} \)
  - Usually from the positive or non-negative integers
  - can be infinite
• \( a_n \) is a term in the sequence
• \( \{a_n\} \) means the entire sequence

Sequence Examples

• \( a_n = 3n \)
  - The terms in the sequence are \( a_1, a_2, a_3, \ldots \)
  - The sequence \( \{a_n\} \) is \{ 3, 6, 9, 12, \ldots \}

• \( b_n = 2^n \)
  - The terms in the sequence are \( b_1, b_2, b_3, \ldots \)
  - The sequence \( \{b_n\} \) is \{ 2, 4, 8, 16, 32, \ldots \}

• Sequences are indexed from 1
  - Not in all textbooks, though!

Geometric vs. Arithmetic Sequences

• The difference is in how they grow
• Arithmetic sequences increase by a constant amount
  - \( a_n = 3n \): \{ 3, 6, 9, 12, \ldots \}
  - Each number is 3 more than the previous
  - Of the form: \( f(x) = dx + a \)

• Geometric sequences increase by a constant factor
  - \( b_n = 2^n \): \{ 2, 4, 8, 16, 32, \ldots \}
  - Each number is twice the previous
  - Of the form: \( f(x) = ar^x \)

Fibonacci sequence

• Sequences can be neither geometric or arithmetic
  - \( F_n = F_{n-1} + F_{n-2} \), where the first two terms are 1
  - Alternative, \( F(n) = F(n-1) + F(n-2) \)
  - Each term is the sum of the previous two terms
  - Sequence: \{ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \}
  - This is the Fibonacci sequence
  - Full formula: \( F(n) = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{\sqrt{5} \cdot 2^n} \)

Fibonacci sequence in nature
Reproducing rabbits

- You have one pair of rabbits on an island
  - The rabbits repeat the following:
    - Get pregnant one month
    - Give birth (to another pair) the next month
  - This process repeats indefinitely (no deaths)
  - Rabbits get pregnant the month they are born

- How many rabbits are there after 10 months?

Reproducing rabbits

- First month: 1 pair
  - The original pair
- Second month: 1 pair
  - The original (and now pregnant) pair
- Third month: 2 pairs
  - The child pair (which is pregnant) and the parent pair (recovering)
- Fourth month: 3 pairs
  - "Grandchildren": Children from the baby pair (now pregnant)
  - Child pair (recovering)
  - Parent pair (pregnant)
- Fifth month: 5 pairs
  - Both the grandchildren and the parents reproduced
  - 3 pairs are pregnant (child and the two new born rabbit pairs)

Fibonacci sequence

- Another application:

Pascal’s Triangle

Fibonacci sequence

- As the terms increase, the ratio between successive terms approaches 1.618
  \[
  \lim_{n \to \infty} \frac{F(n+1)}{F(n)} = q = \frac{\sqrt{5} + 1}{2} = 1.61803398874... = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}
  \]

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell
The Golden Ratio

Determining the sequence formula

- Given values in a sequence, how do you determine the explicit formula?
- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

- 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, ...  
  - alternates 1’s and 0’s, increasing the number of 1’s and 0’s each time
- 1, 2, 2, 3, 4, 5, 6, 6, 7, 8, 8, ... 
  - increases by one, but repeats all even numbers once
- 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ...  
  - non-0 numbers are a geometric sequence ($2^n$) interspersed with zeros
- 3, 6, 12, 24, 48, 96, 192, ...  
  - Each term is twice the previous: geometric progression
  - $a_n = 3 \cdot 2^{n-1}$

- 15, 8, 1, -6, -13, -20, -27, ...  
  - Each term is 7 less than the previous term
  - $a_n = 22 - 7n$
- 3, 5, 8, 12, 17, 23, 30, 38, 47, ...  
  - The difference between successive terms increases by one each time: $a_1 = 3$, $a_n = a_{n-1} + n$
  - $a_n = n(n+1)/2 + 2$
- 2, 16, 54, 128, 250, 432, 686, ...  
  - Each term is twice the cube of $n$
  - $a_n = 2 \cdot n^3$
- 2, 3, 7, 25, 121, 721, 5041, 40321 
  - Each successive term is about $n$ times the previous
  - $a_n = n! + 1$

Summations

- A summation:

\[
\sum_{i=m}^{n} a_i \quad \text{or} \quad \sum_{i=m}^{n} a_i
\]

- is like a for loop:

```python
sum := 0
for (i := m to n)
    sum := sum + a_i
next i
```

Evaluating sequences

- $\sum_{i=1}^{k} (2i+1)$  
  - $2 + 3 + 4 + 5 + 6 = 20$
- $\sum_{i=0}^{n} (-2)^i$  
  - $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$
- $\sum_{i=3}^{3} 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$
- $\sum_{i=3}^{5} 2^{i-1} - 2^i$  
  - $(2^1 - 2^2) + (2^2 - 2^3) + (2^3 - 2^4) + (2^4 - 2^5) = 511$
  - Note that each term (except the first and last) is cancelled by another term
More Notations

• Product: \[ \prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots \times a_n \]

• Factorial: \[ n! = n \times (n - 1) \times \ldots \times 3 \times 2 \times 1 \]

• \( n \) choose \( r \): \[ \binom{n}{r} = \frac{n!}{r!(n-r)!} \]

Properties

\[ \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} (a_i + b_i) \]

\[ c \sum_{i=m}^{n} a_i = \sum_{i=m}^{n} (c \times a_i) \]

\[ \prod_{i=m}^{n} a_i \times \prod_{i=m}^{n} b_i = \prod_{i=m}^{n} (a_i \times b_i) \]

Double summations

• Like a nested for loop

\[ \sum_{i=1}^{4} \sum_{j=1}^{3} ij \]

• Is equivalent to:

```
int sum = 0;
for (int i=1; i<=4; i++)
    for (int j=1; j<=3; j++)
        sum += i*j;
```