Review

• \( p \rightarrow q \equiv (p \lor q) \land \neg(p \land q) \)
• \( \neg p \lor q \equiv \neg(p \land q) \equiv \neg p \lor \neg q \)
• Contrapositive:
  • \( \neg q \rightarrow \neg p \equiv \neg(p \lor q) \equiv \neg p \land \neg q \)
• Inverse:
  • \( \neg p \rightarrow \neg q \)
  • \( p \text{ is sufficient for } q \equiv p \rightarrow q \)
  • \( p \text{ is necessary for } q \)
• Converse:
  • \( q \rightarrow p \equiv \neg q \lor p \equiv \neg(p \land q) \equiv \neg p \lor \neg q \)

Valid and Invalid Arguments

CS 231
Dianna Xu

Definitions

• An argument is a sequence of statements (statement forms).
• All statements in an argument except for the last one, are called premises.
  (assumptions, hypotheses)
• The final statement is the conclusion.
• A valid argument means the conclusion is true if the premises are all true, with all combinations of variable truth values.

Examples

• All Greeks are human and all humans are mortal; therefore, all Greeks are mortal.
• Some men are athletes and some athletes are rich; therefore, some men are rich.
• Some men are swimmers and some swimmers are fish; therefore, some men are fish.

Modus Ponens

\[
p \\
p \rightarrow q \\
\therefore q
\]
Modus Ponens example

• Assume you are given the following two statements:
  – "you are in this class" \( p \)
  – "If you are in this class, you are a student" \( p \implies q \)

• Let \( p = \) "you are in this class"
• Let \( q = \) "you are a student"

• By Modus Ponens, you can conclude that you are a student.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{p} & \text{q} & \text{p \implies q} & \text{(p \land (p \implies q)) \implies q} \\
\hline
T & T & T & T \\
T & F & F & T \\
F & T & F & T \\
F & F & F & T \\
\hline
\end{array}
\]

Modus Ponens

• Consider \((p \land (p \implies q)) \implies q\)

Modus Tollens example

• Assume that we know: \(\neg q\) and \(p \implies q\)
  – Recall that \(p \implies q \equiv \neg q \implies \neg p\)
• Thus, we know \(\neg q\) and \(\neg q \implies \neg p\)
• We can conclude \(\neg p\)

\[
\begin{align*}
\neg q \\
p \implies q \\
\therefore \neg p
\end{align*}
\]

Generalization & Specialization

• Generalization: If you know that \(p\) is true, then \(p \lor q\) will ALWAYS be true.

\[
\begin{array}{|c|c|}
\hline
p & q \\
\hline
\therefore p \lor q & \therefore p \lor q \\
\hline
\end{array}
\]

• Specialization: If \(p \land q\) is true, then \(p\) will ALWAYS be true.

\[
\begin{array}{|c|c|}
\hline
p & q \\
\hline
\therefore p & \therefore q \\
\hline
\end{array}
\]

Example of proof

• We have the hypotheses:
  \(p\) – "it is not sunny this afternoon and it is colder than yesterday"
  \(q\) – "We will go swimming only if it is sunny"
  \(r\) – "If we do not go swimming, then we will take a canoe trip"
  \(s\) – "If we take a canoe trip, then we will be home by sunset"
  \(t\) – "We will be home by sunset"

• Does this imply that "we will be home by sunset"?
Example of proof

1. \( \neg p \land q \)  1st hypothesis
2. \( \neg p \) Specialization using step 1
3. \( r \rightarrow p \)  2nd hypothesis
4. \( \neg r \rightarrow s \)  3rd hypothesis
5. \( s \rightarrow t \) Modus ponens using steps 4 & 5
6. \( t \rightarrow p \)  4th hypothesis
7. \( p \rightarrow q \)  5th hypothesis
8. \( p \land q \rightarrow q \)  3rd hypothesis
9. \( p \rightarrow q \)  6th hypothesis
10. \( s \rightarrow t \)  7th hypothesis
11. \( r \rightarrow s \) Transitivity using steps 1, 2, & 3
12. \( p \rightarrow q \)  8th hypothesis
13. \( q \rightarrow r \)  9th hypothesis
14. \( t \rightarrow p \)  10th hypothesis
15. \( p \land q \rightarrow q \)  11th hypothesis
16. \( p \land q \rightarrow q \)  12th hypothesis
17. \( p \land q \rightarrow q \)  13th hypothesis
18. \( p \land q \rightarrow q \)  14th hypothesis

More rules of inference

- Conjunction: if \( p \) and \( q \) are true separately, then \( p \land q \) is true
- Elimination: If \( p \lor q \) is true, and \( p \) is false, then \( q \) must be true
- Transitivity: If \( p \rightarrow q \) is true, and \( q \rightarrow r \) is true, then \( p \rightarrow r \) must be true

Even more rules of inference

- Proof by division into cases: if at least one of \( p \) or \( q \) is true, then \( r \) must be true
- Contradiction rule: If \( \neg p \rightarrow c \) is true, we can conclude \( p \) (via the contra-positive)
- Resolution: If \( p \lor q \) is true, and \( \neg p \lor r \) is true, then \( q \lor r \) must be true

Example of proof

1. \( \neg t \)  3rd hypothesis
2. \( s \rightarrow t \)  2nd hypothesis
3. \( \neg s \) Modus tollens using steps 1 & 2
4. \( \neg \neg t \rightarrow \neg (s \land t) \) 1st hypothesis
5. \( \neg (s \land t) \rightarrow \neg \neg t \) Contrapositive of step 4
6. \( \neg (s \land t) \rightarrow \neg (s \land t) \) DeMorgan’s law and double negation law
7. \( \neg q \rightarrow r \) Generalization using step 3
8. \( r \rightarrow q \)  4th hypothesis
9. \( r \rightarrow q \)  5th hypothesis
10. \( r \rightarrow q \)  6th hypothesis
11. \( r \rightarrow q \)  7th hypothesis
12. \( q \rightarrow r \)  8th hypothesis
13. \( q \rightarrow r \)  9th hypothesis
14. \( q \rightarrow r \)  10th hypothesis

Example of proof

1. \( \neg t \)  3rd hypothesis
2. \( s \rightarrow t \)  2nd hypothesis
3. \( \neg s \) Modus tollens using steps 1 & 2
4. \( \neg \neg t \rightarrow \neg (s \land t) \) 1st hypothesis
5. \( \neg (s \land t) \rightarrow \neg \neg t \) Contrapositive of step 4
6. \( \neg (s \land t) \rightarrow \neg (s \land t) \) DeMorgan’s law and double negation law
7. \( \neg q \rightarrow r \) Generalization using step 3
8. \( r \rightarrow q \)  4th hypothesis
9. \( r \rightarrow q \)  5th hypothesis
10. \( r \rightarrow q \)  6th hypothesis
11. \( r \rightarrow q \)  7th hypothesis
12. \( q \rightarrow r \)  8th hypothesis
13. \( q \rightarrow r \)  9th hypothesis
14. \( q \rightarrow r \)  10th hypothesis

Example of proof

1. \( \neg q \)  3rd hypothesis
2. \( q \rightarrow r \)  2nd hypothesis
3. \( \neg r \)  1st hypothesis
4. \( \neg q \rightarrow r \) Generalization using step 3
5. \( \neg q \rightarrow r \)  4th hypothesis
6. \( \neg q \rightarrow r \)  5th hypothesis
7. \( \neg q \rightarrow r \)  6th hypothesis
8. \( \neg q \rightarrow r \)  7th hypothesis
9. \( \neg q \rightarrow r \)  8th hypothesis
10. \( \neg q \rightarrow r \)  9th hypothesis
11. \( \neg q \rightarrow r \)  10th hypothesis

Example of proof

1. \( \neg t \)  3rd hypothesis
2. \( s \rightarrow t \)  2nd hypothesis
3. \( \neg s \) Modus tollens using steps 1 & 2
4. \( \neg \neg t \rightarrow \neg (s \land t) \) 1st hypothesis
5. \( \neg (s \land t) \rightarrow \neg \neg t \) Contrapositive of step 4
6. \( \neg (s \land t) \rightarrow \neg (s \land t) \) DeMorgan’s law and double negation law
7. \( \neg q \rightarrow r \) Generalization using step 3
8. \( r \rightarrow q \)  4th hypothesis
9. \( r \rightarrow q \)  5th hypothesis
10. \( r \rightarrow q \)  6th hypothesis
11. \( r \rightarrow q \)  7th hypothesis
12. \( q \rightarrow r \)  8th hypothesis
13. \( q \rightarrow r \)  9th hypothesis
14. \( q \rightarrow r \)  10th hypothesis

Modus Badus

- Consider the following:
- Is this true?

Fallacy of the converse

Fallacy of affirming the conclusion

Not a valid rule!
Modus Badus example

- Assume you are given the following two statements:
  - “you are a student”
  - “if you are in this class, you are a student”
- Let $p$ = “you are in this class”
- Let $q$ = “you are a student”
- It is clearly wrong to conclude that if you are a student, you must be in this class

Fallacy of the inverse - Modus Badus - Fallacy of denying the hypothesis

- Consider the following:
  - $p \rightarrow q$
  - $\neg p$
- Is this true?

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p \land (p \rightarrow q)$</th>
<th>$(\neg p \land (p \rightarrow q)) \rightarrow \neg q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Not a valid rule!

---

Modus Badus example

- Assume you are given the following two statements:
  - “you are not in this class”
  - “if you are in this class, you are a student”
- Let $p$ = “you are in this class”
- Let $q$ = “you are a student”
- You CANNOT conclude that you are not a student just because you are not taking Discrete Math