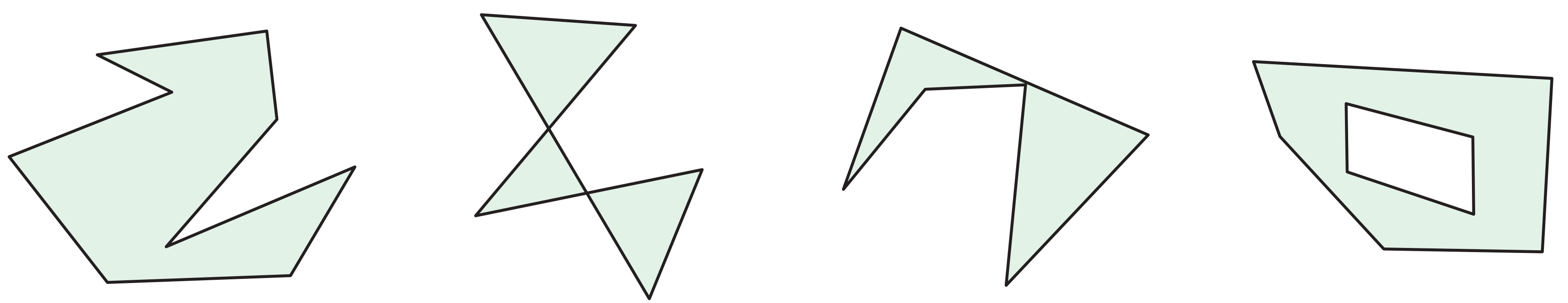
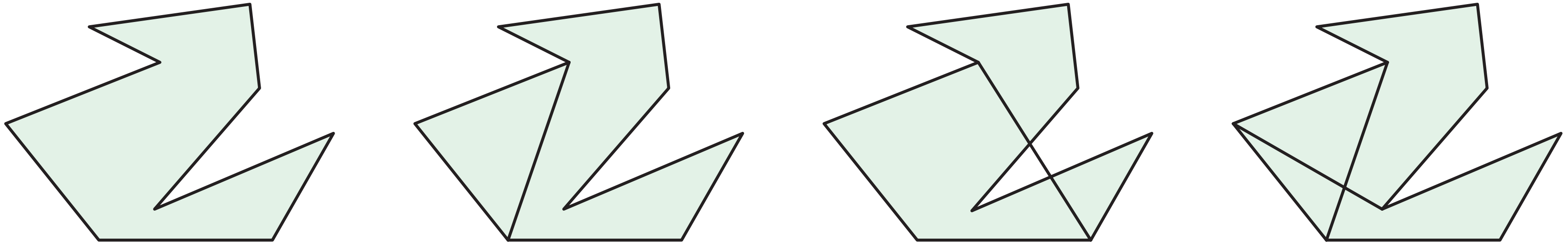
Triangulations and MST

Polygon

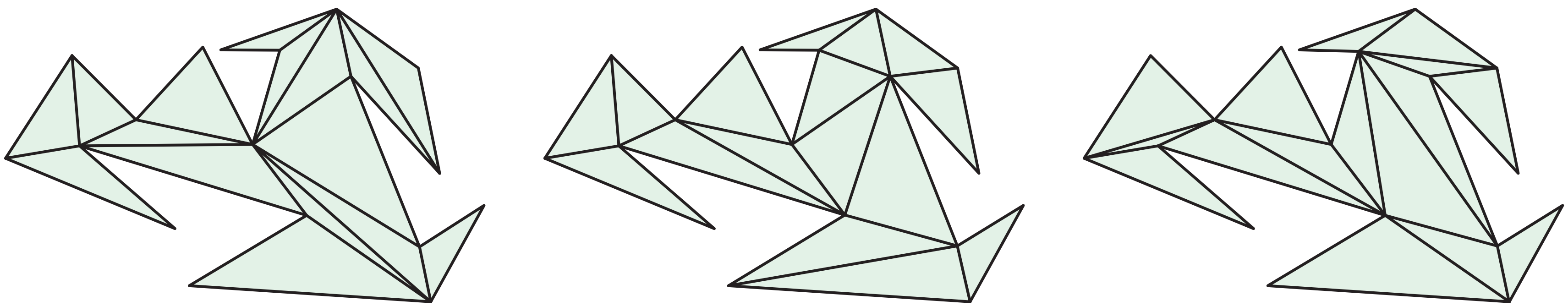


Diagonal



Triangulation

- A triangulation of a polygon is a decomposition into triangles with maximal non-crossing diagonals.



- Every polygon of n>3 vertices has at least one diagonal

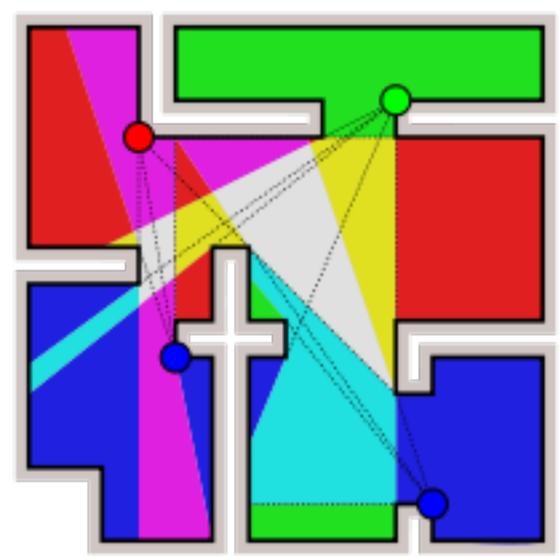
Theorem

- Every polygon admits a triangulation.
- Proof by strong induction

- Every triangulation of a polygon P with n vertices has n-2 triangles and n-3 diagonals.
- Proof by strong induction

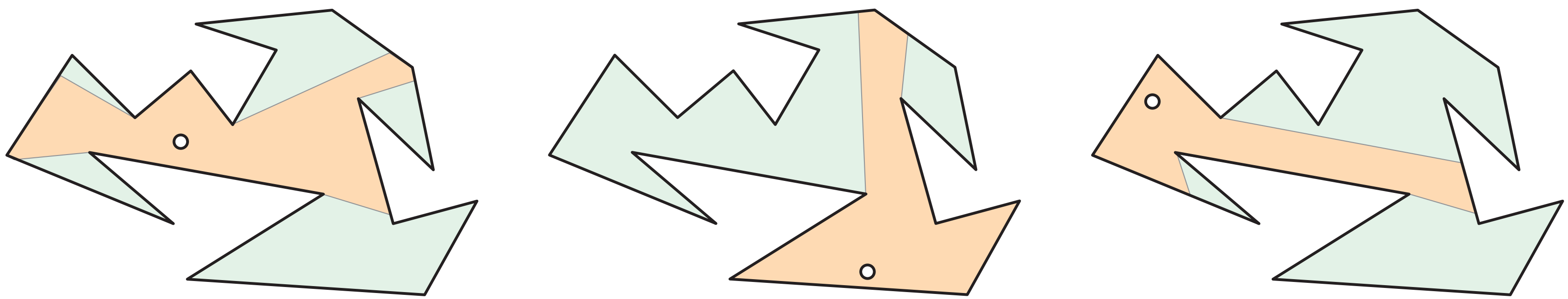
The Art Gallery Problem

- Polygon models the floor plan
- Guards are stationary and have 360° visibility unless blocked by walls
- What is the minimum number of guards needed to cover an arbitrary polygon of *n* vertices?

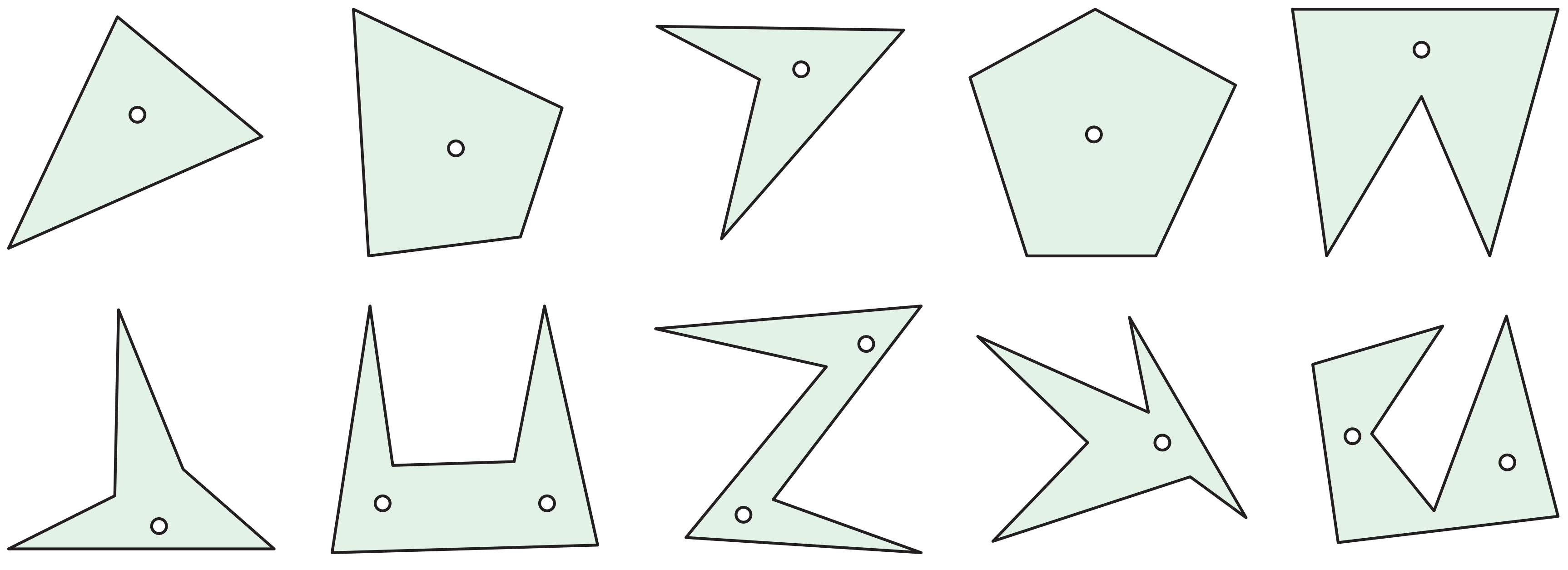


Visibility

- Vertices do not block vision
- xy∈P → x sees y

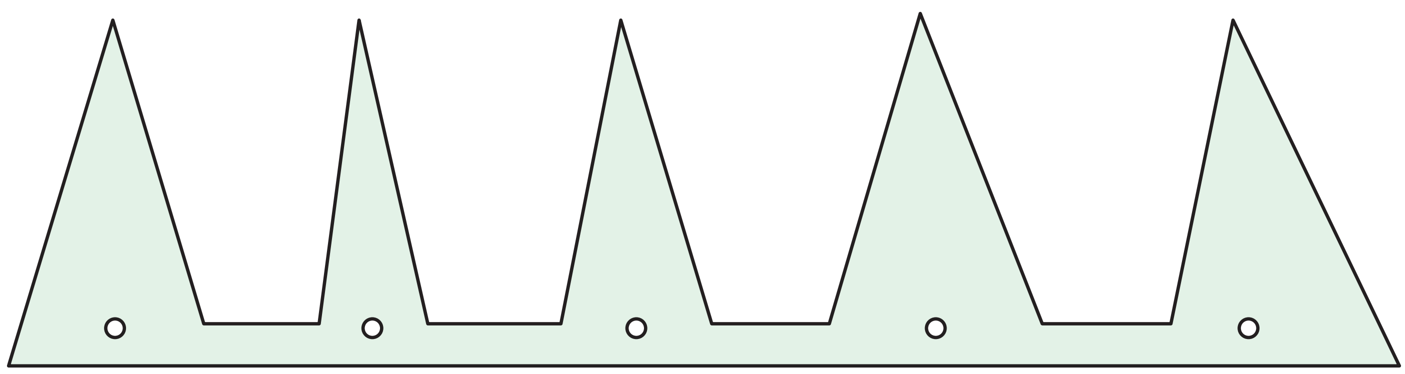


Examples



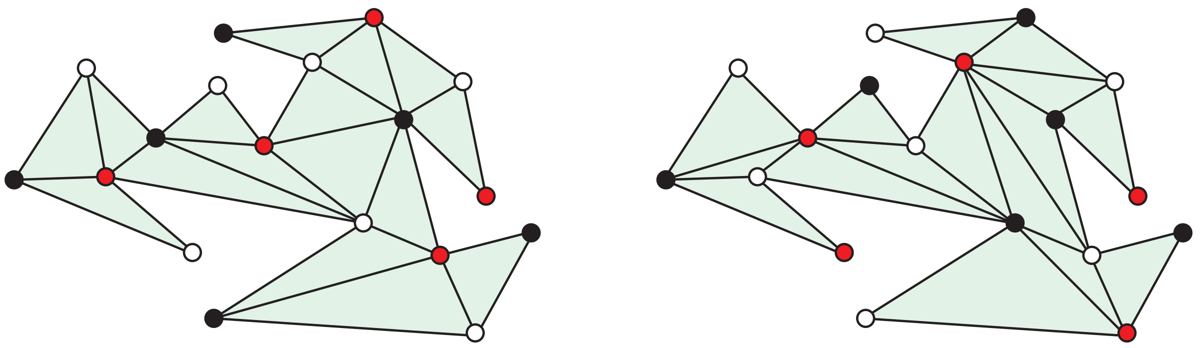
The Necessity of $\lfloor n/3 \rfloor$

- The comb



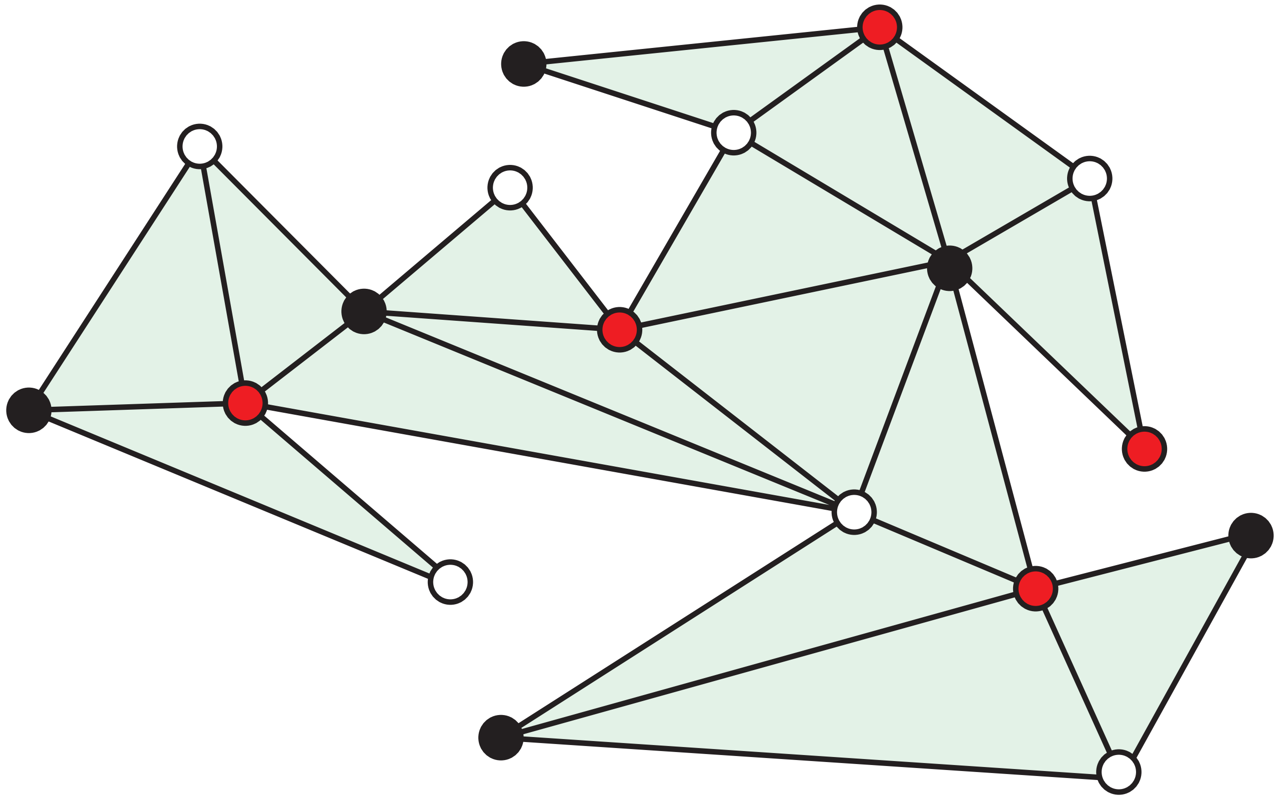
The Sufficiency of $\lfloor n/3 \rfloor$

- A coloring of a graph is an assignment of colors to nodes so that no adjacent nodes have the same color

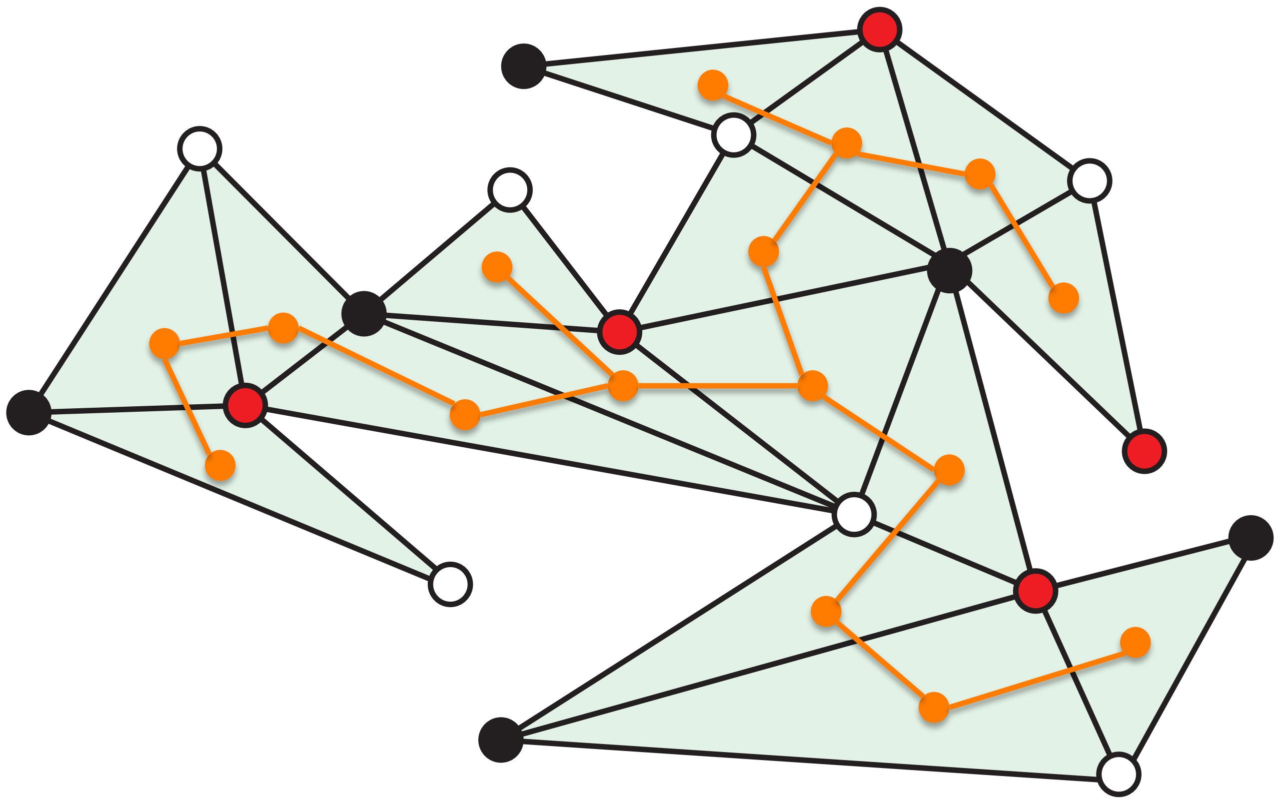


Every Planar Graph Can be 4-colored

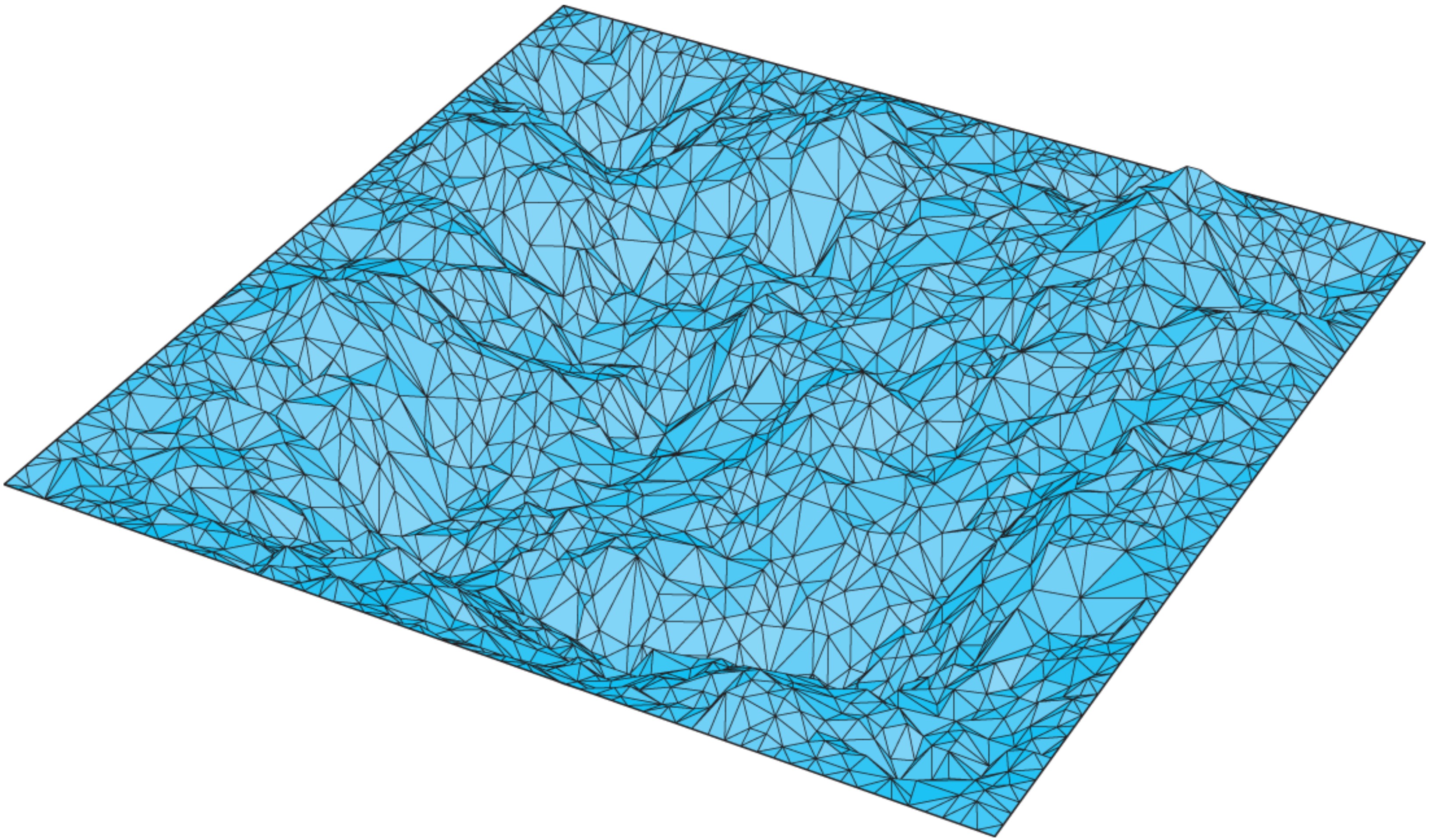
Every Triangulation Can be 3-colored



Dual Graph

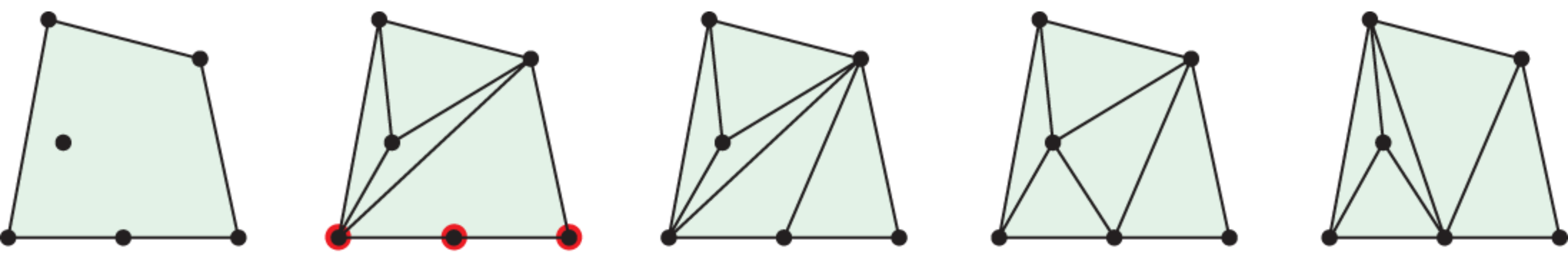


Terrain Reconstruction from Sampled Heights



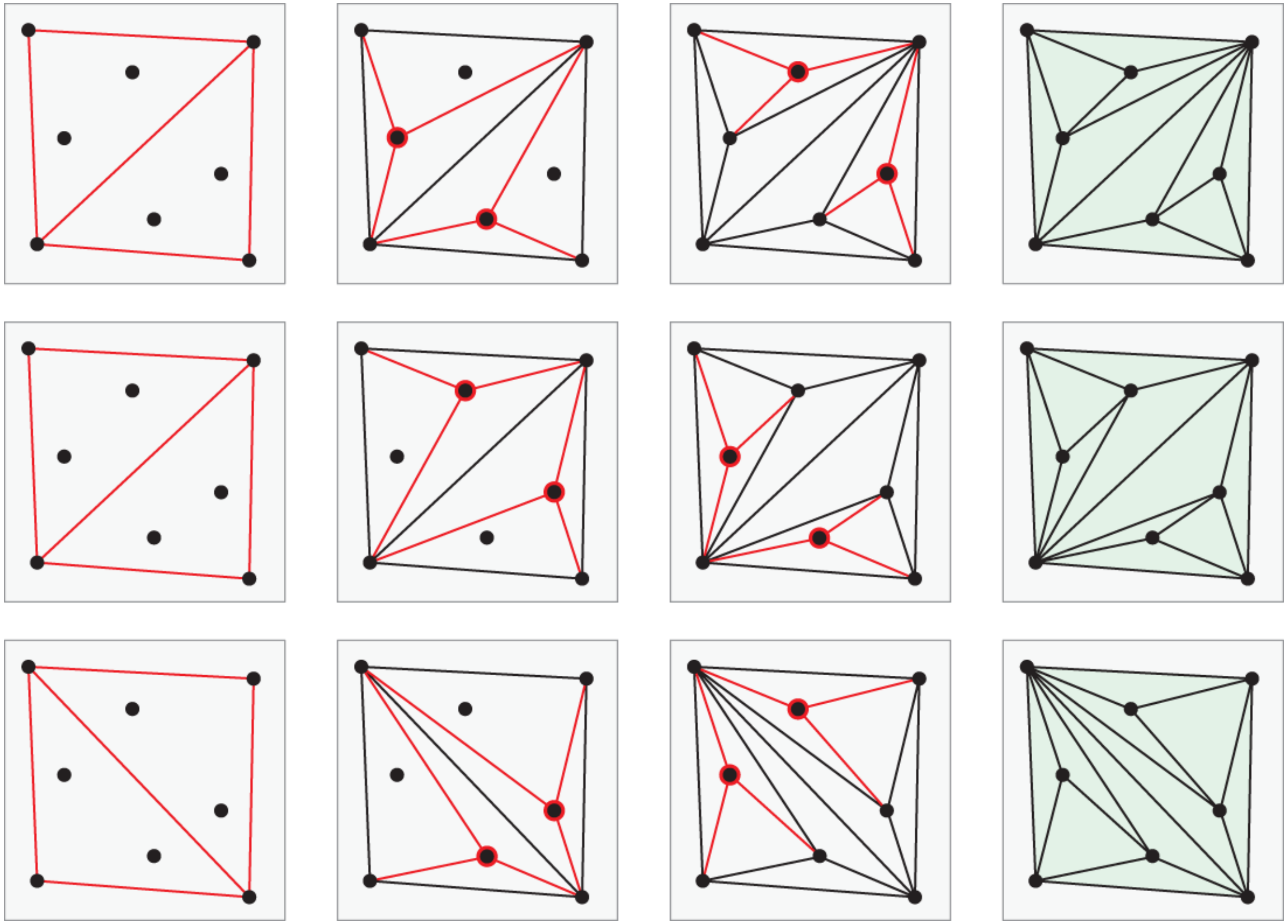
Definition

- A triangulation of a planar point set *S* is a subdivision of the plane determined by a *maximal* set of non-crossing edges whose vertex set is *S*.

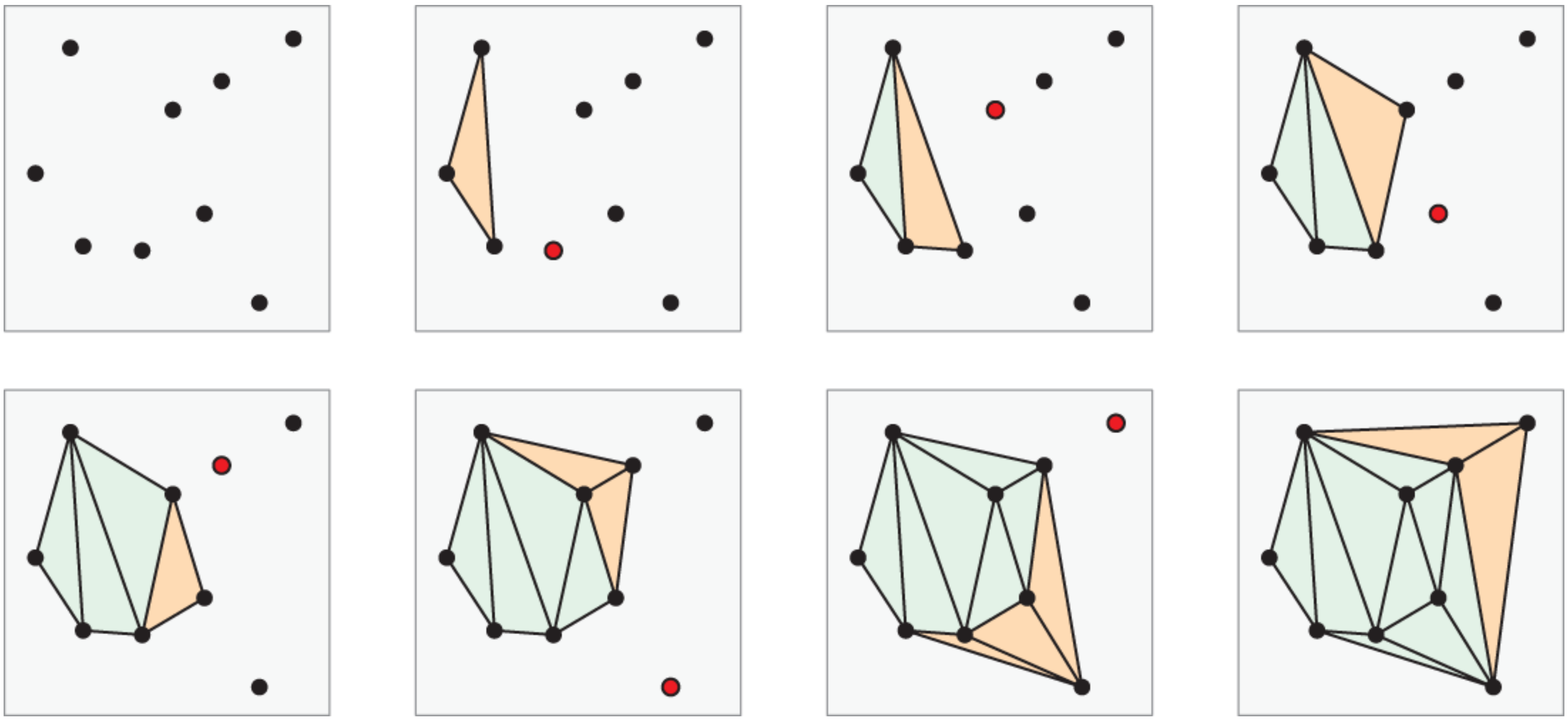


(a) (b) (c) (d) (e)

Triangle Splitting

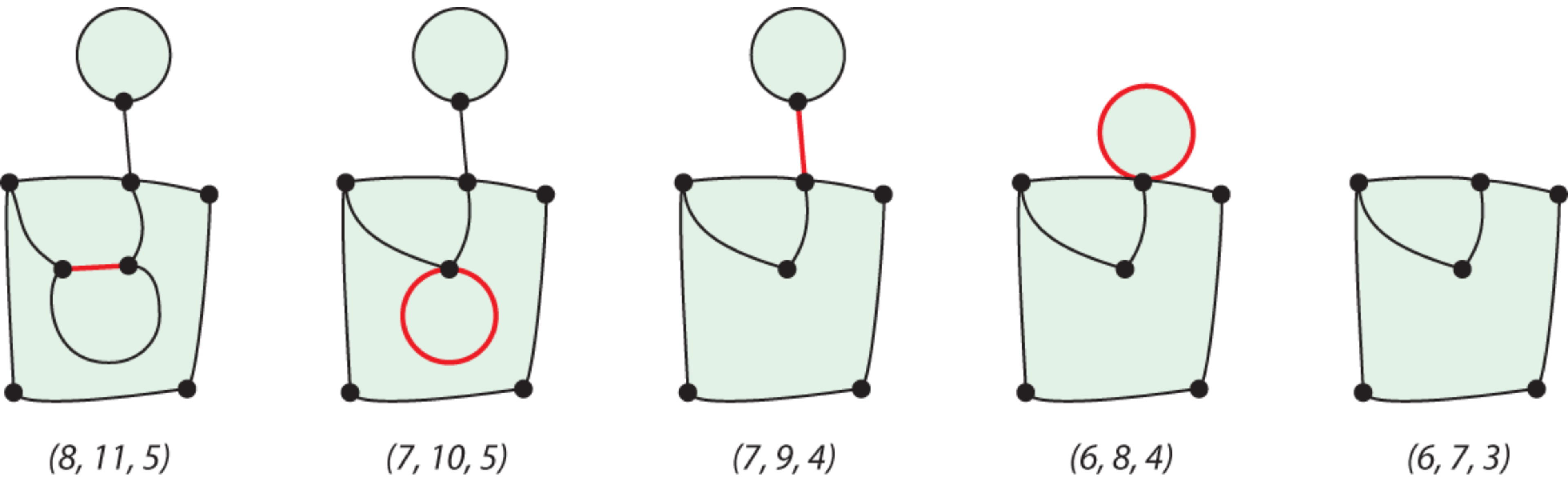


Incremental



Euler’s Formula

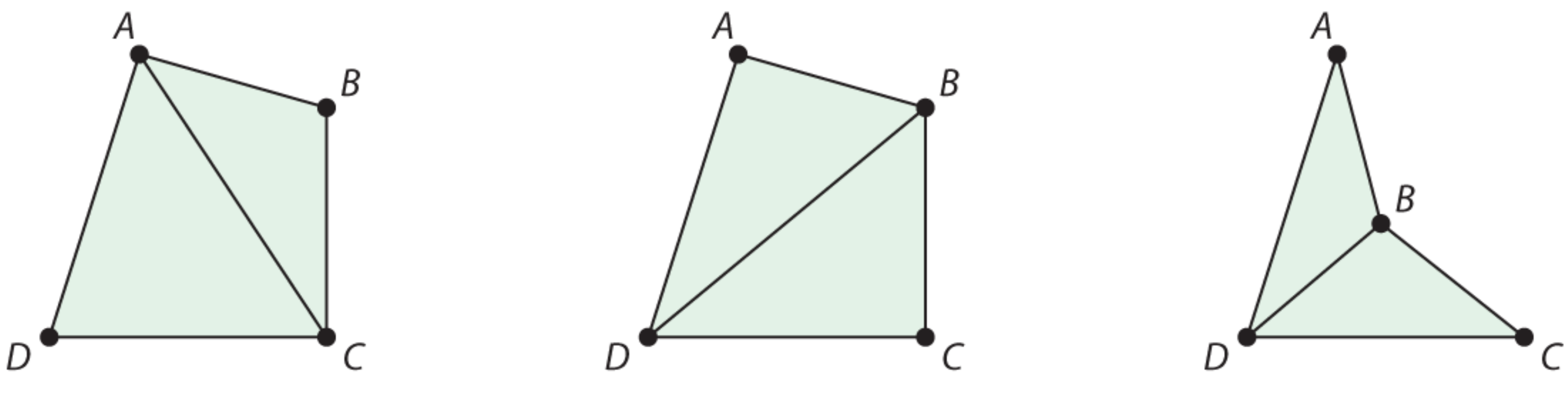
- Let G be a connected planar graph with V vertices, E edges and F faces, then V-E+F = 2
- The outer face is unbounded
- Proof by induction on the number of edges



Theorem

- Let S be a point set with h points on the hull and k in the interior. If all points are in general position, then any triangulation of S has exactly $2k+h-2$ triangles and $3k+2h-1$ edges.
- Proof

Edge Flip

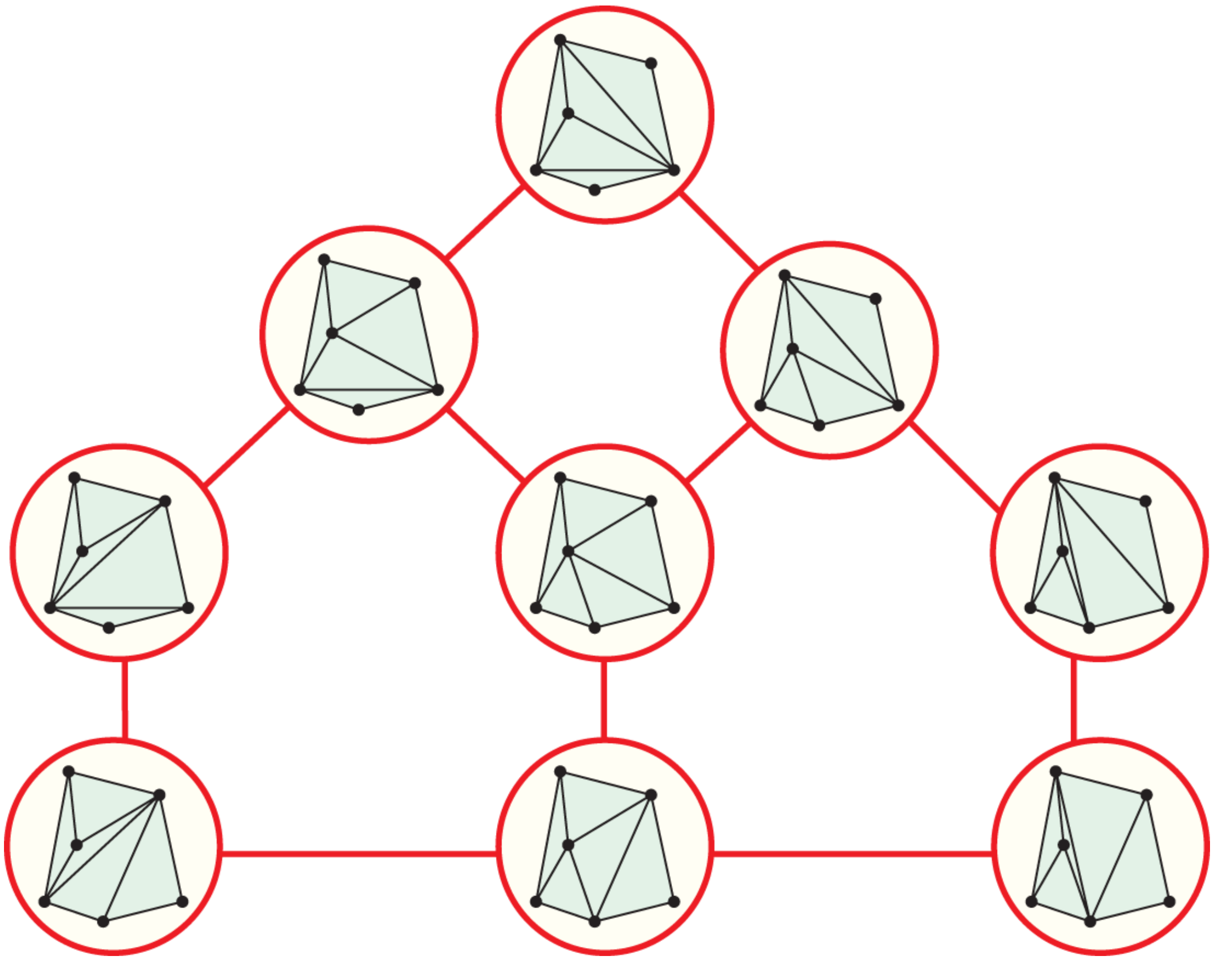


(a) (b) (c)

Definition

- For a point set S, a flip graph of S is a graph whose nodes are the set of triangulations of S. Two nodes T_1 and T_2 are connected by an edge if one diagonal of T_1 can be flipped to obtain T_2.

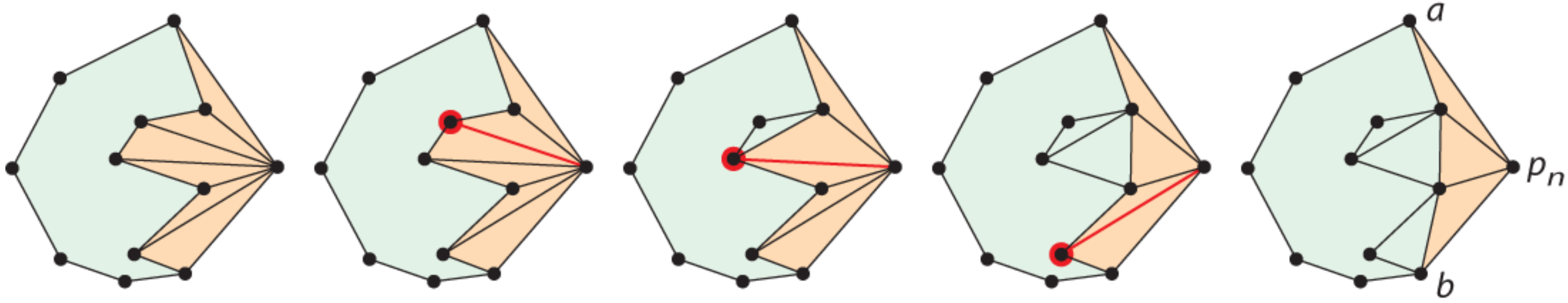
Flip Graph



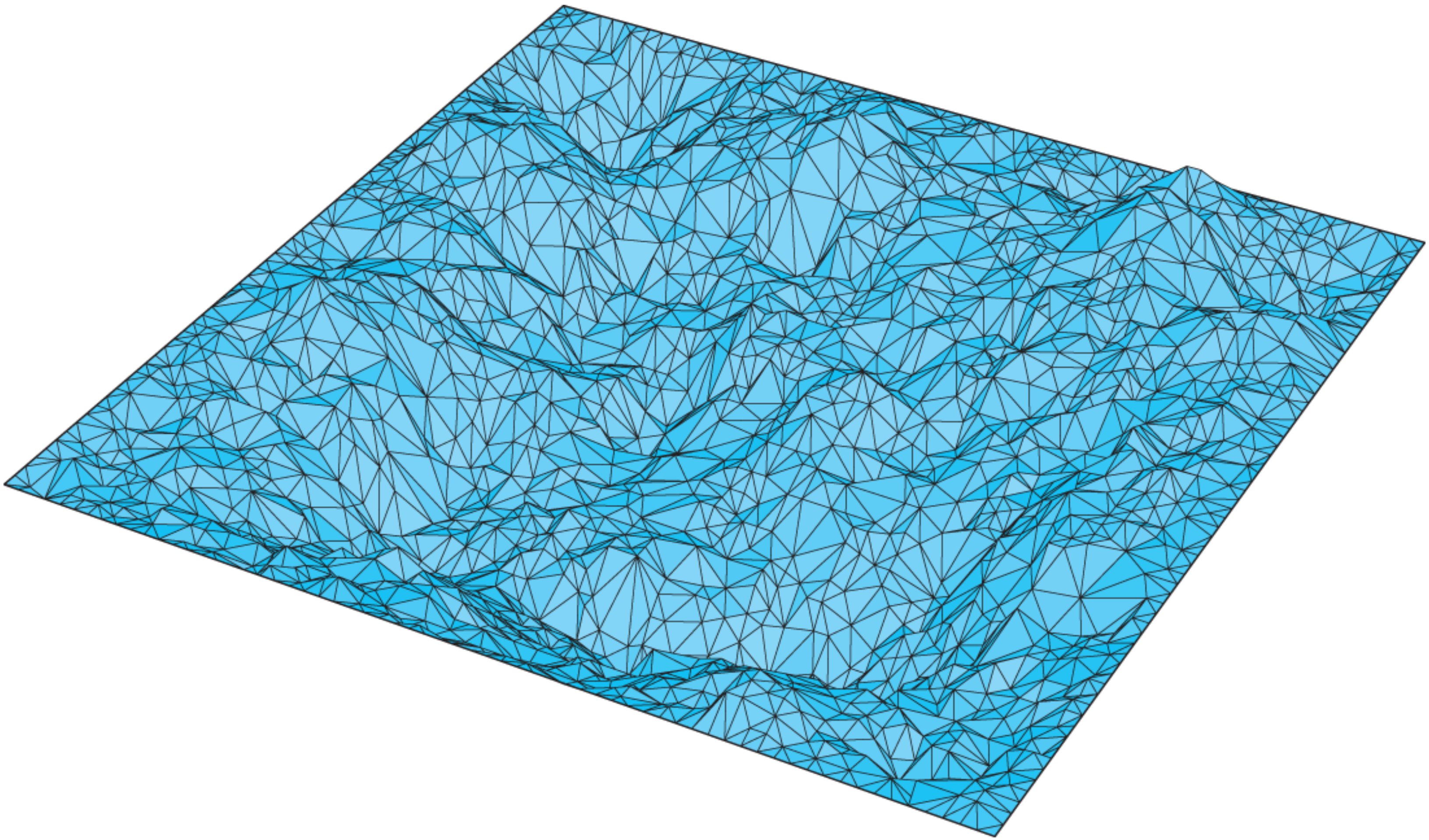
Theorem

- The flip graph of any planar point set is connected.
- Proof by induction

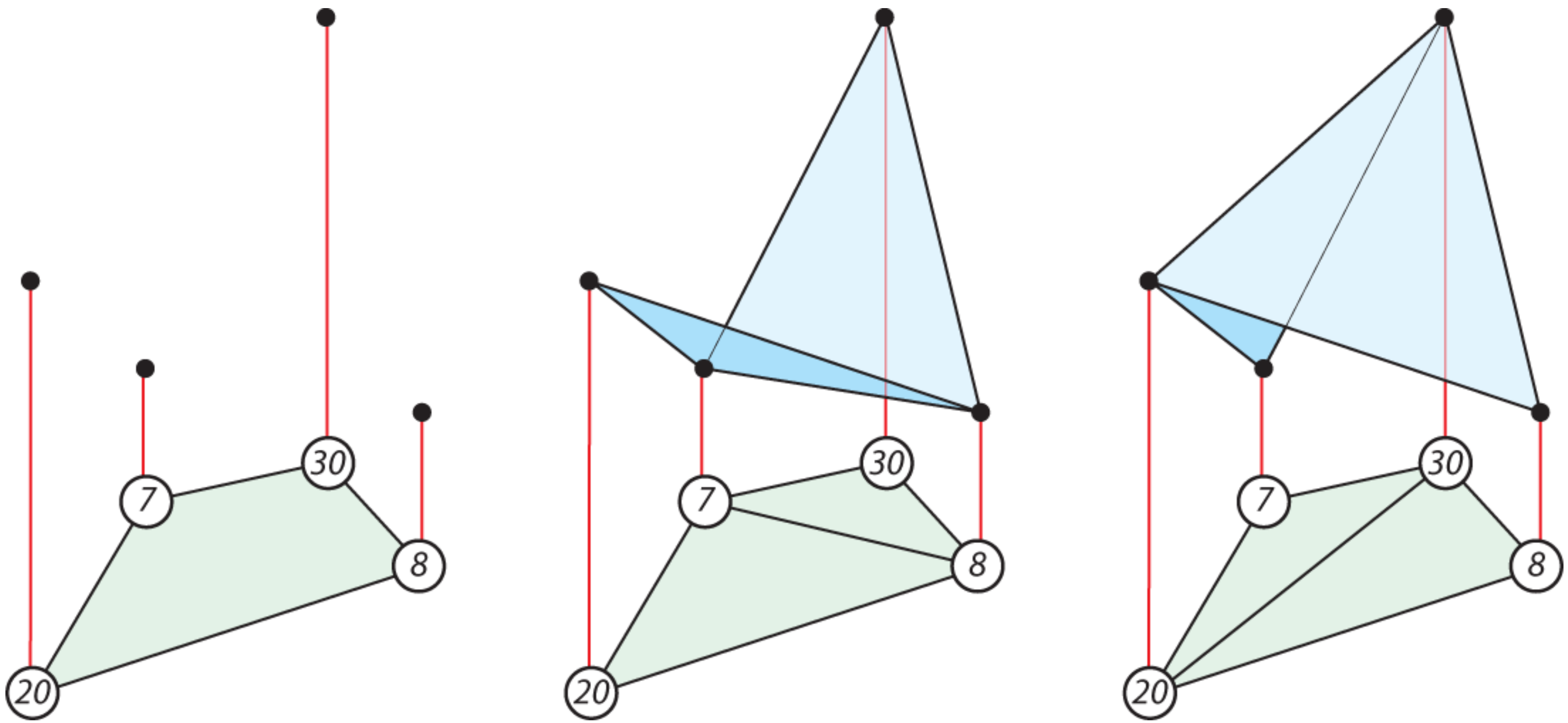
Flipping a Star



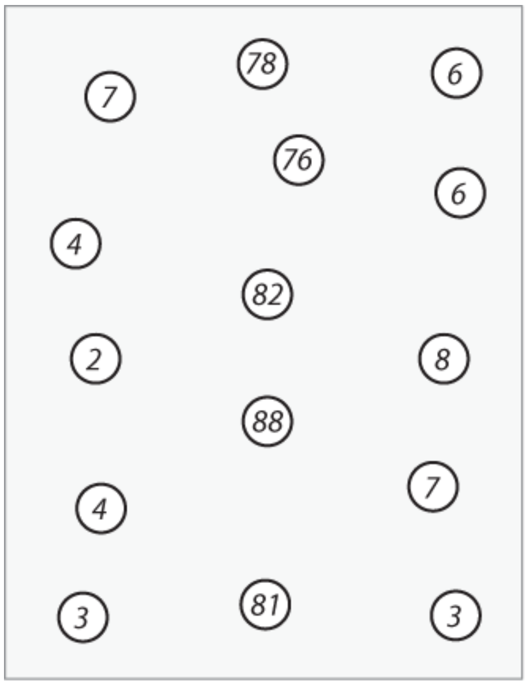
3D Terrain from Sampled Points



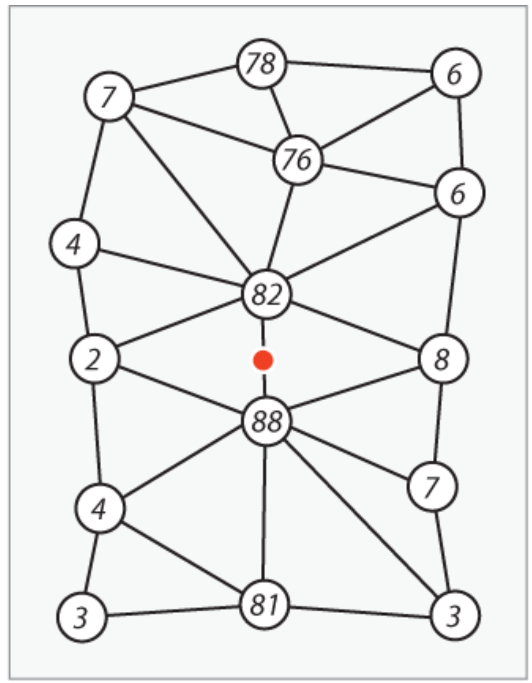
Lifting the Triangles



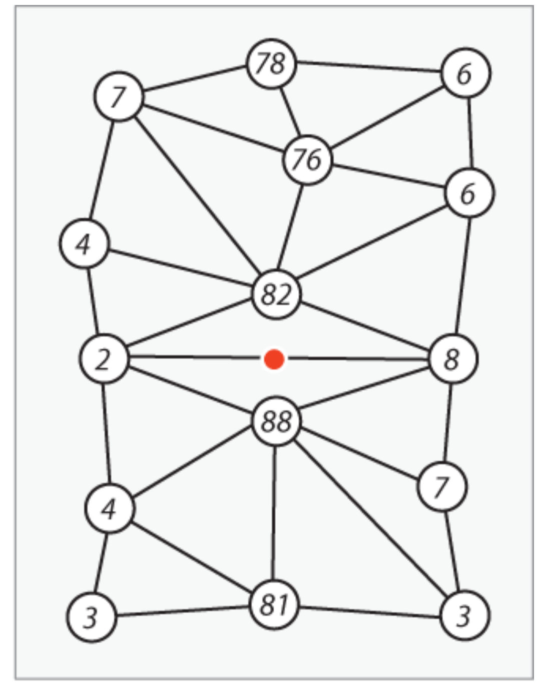
Skinny is Bad



(a)



(b)



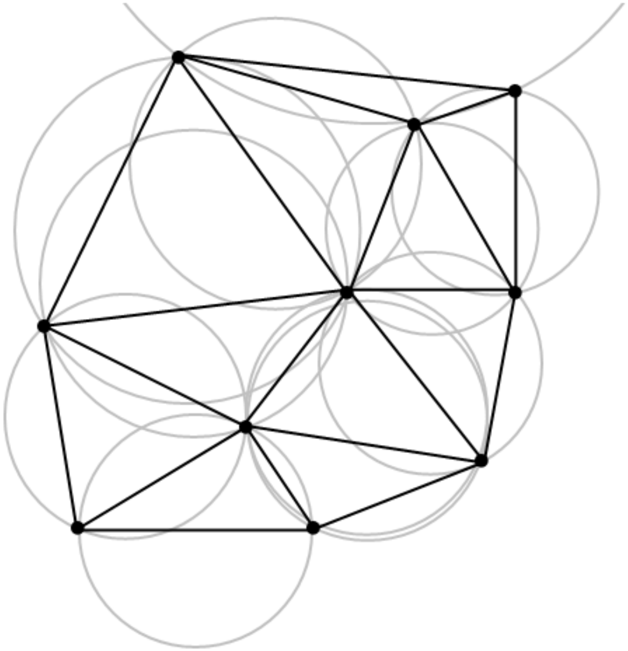
(c)

Angle Sequence

- Let T be a triangulation of a point set S, and suppose T has n triangles. The angle sequence $\{a_1, a_2, \ldots, a_n\}$ lists all $3n$ angles of T in sorted order.
- A triangulation T_1 is fatter than T_2 ($T_1 > T_2$) if the angle sequence of T_1 is lexicographically greater than T_2's.
  - $\{20°, 30°, 45°, 65°, 120°\} > \{20°, 30°, 45°, 60°, 120°\}$

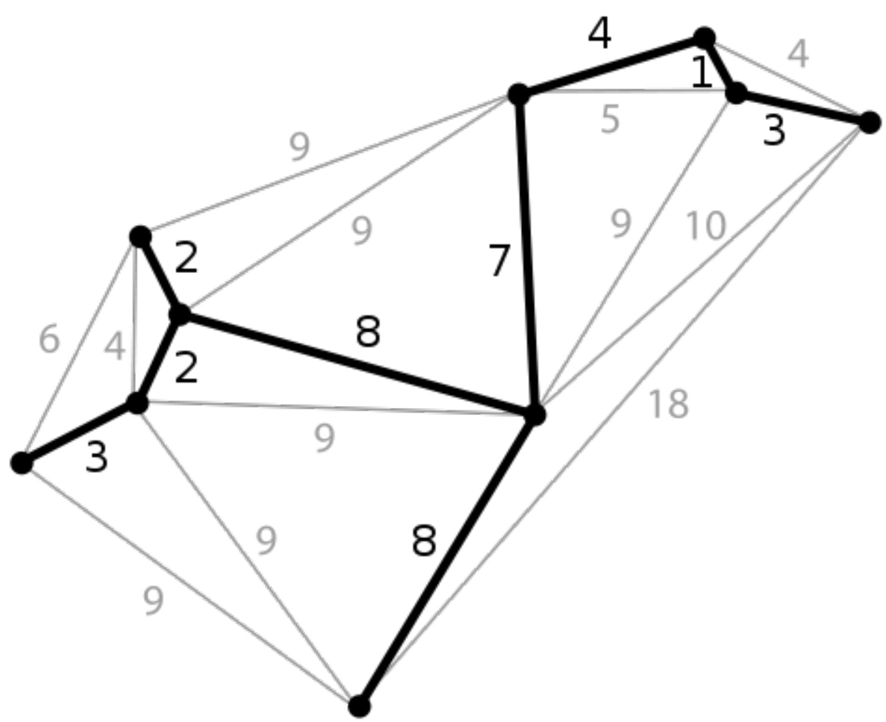
Delaunay Triangulation

- For each convex quad in a triangulation T_1 with diagonal *e*, if a diagonal flip results in a triangulation T_2, s.t. $T_1 \geq T_2$, then *e* is legal.
- A Delaunay triangulation is a triangulation with all legal edges.



When Edges Have Weights

- A minimum spanning tree (MST) of a graph is a tree that connects every vertex and minimizes the total edge weights (lengths).



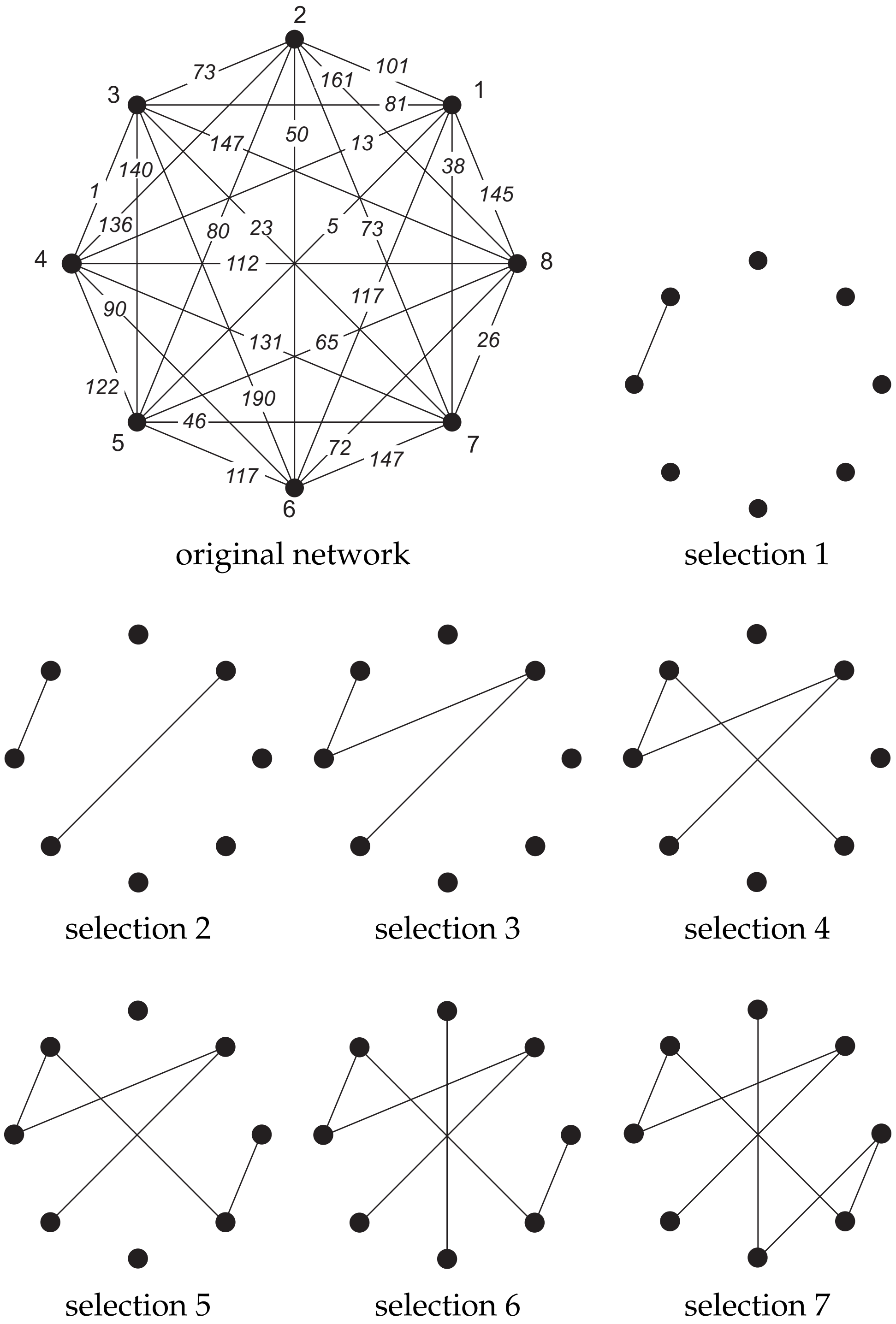
Two Greedy Algorithms

- Kruskal's: An algorithm that always chooses the next shortest edge that does not result in a cycle.

- Prim's: Similar, but maintains a connected tree at all times
  - start with $V_{MST} = \{v_x\}$ and $E_{MST} = \{\}$
  - repeat until $V_{MST} = V$: find min $e = \{v_i, v_j\}$ such that v_i is in V_{MST} and v_j is not. Add v_j to V_{MST} and add e to E_{MST}

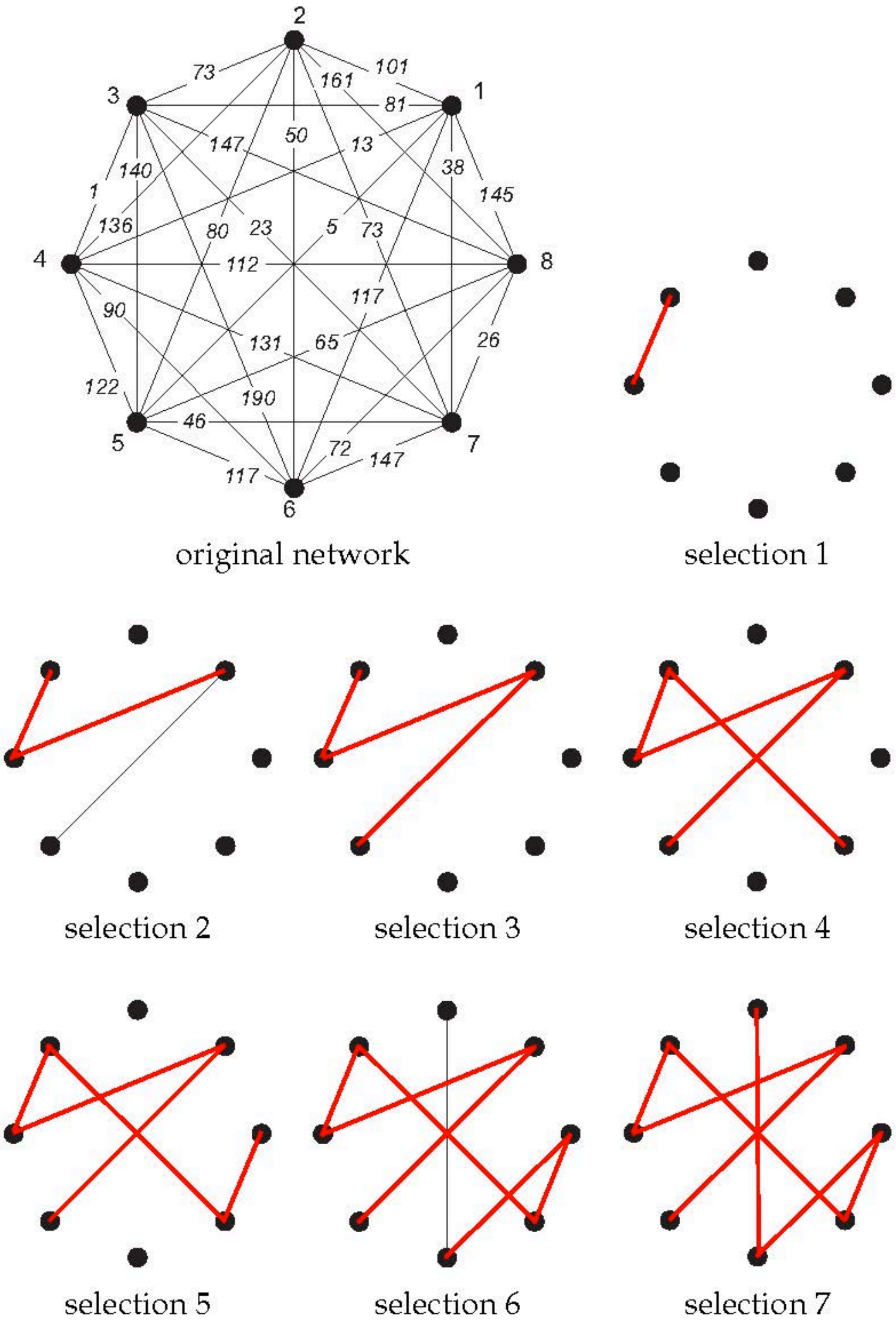
Kruskal's

Edge	Weight	Comment
(3, 4)	1	selection 1
(1, 5)	5	selection 2
(1, 4)	13	selection 3
(3, 7)	23	selection 4
(7, 8)	26	selection 5
(1, 7)	38	cycle (1,7,3,4,1)
(5, 7)	46	cycle (1,5,7,3,4,1)
(2, 6)	50	selection 6
(5, 8)	65	cycle (1,5,8,7,3,4,1)
(6, 8)	72	selection 7



Prim's

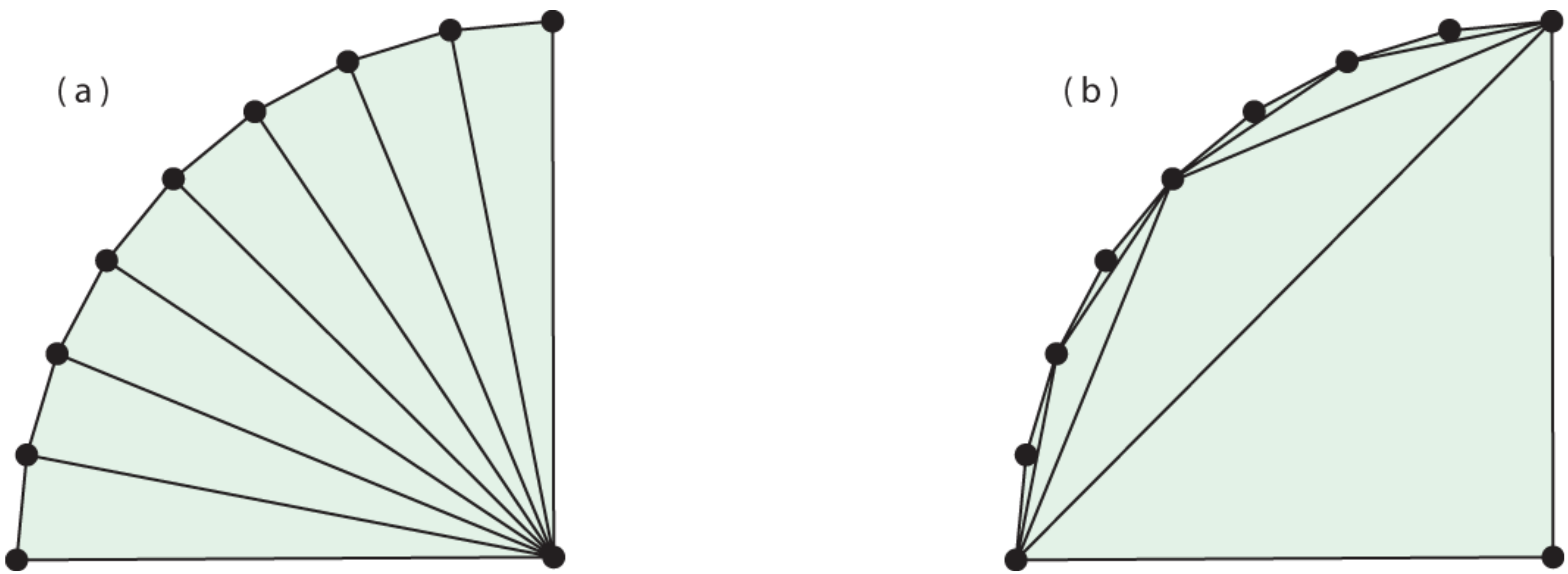
Vertex	Edge	Weight	Comment
4			selection 0
3	(3, 4)	1	selection 1
1	(1, 4)	13	selection 2
5	(1, 5)	5	selection 3
7	(3, 7)	23	selection 4
8	(7, 8)	26	selection 5
6	(6, 8)	72	selection 6
2	(2, 6)	50	selection 7



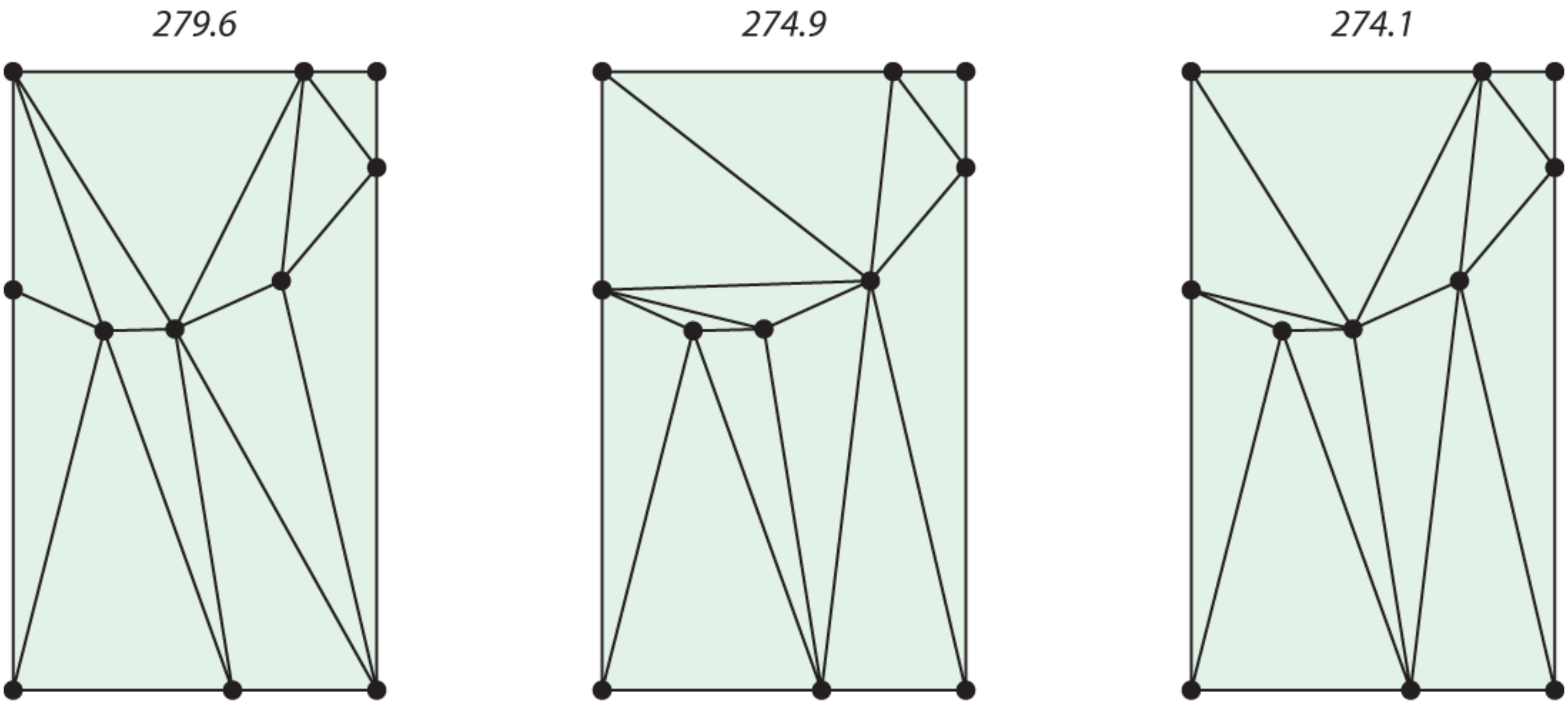
Minimum Weight Triangulation

- A minimum weight triangulation (MWT) is a triangulation of a point set that minimizes the total edge lengths (weights).

Delaunay is not MWT



Delaunay vs. Greedy vs. MWT



(a) (b) (c)

Theorem

- For point set *S*, a minimum spanning tree of *S* is a subset of the Delaunay triangulation of *S*.
- Proof by contradiction.