THE WEIGHTED SHORTEST PATH PROBLEM

Weighted Shortest Path Problem

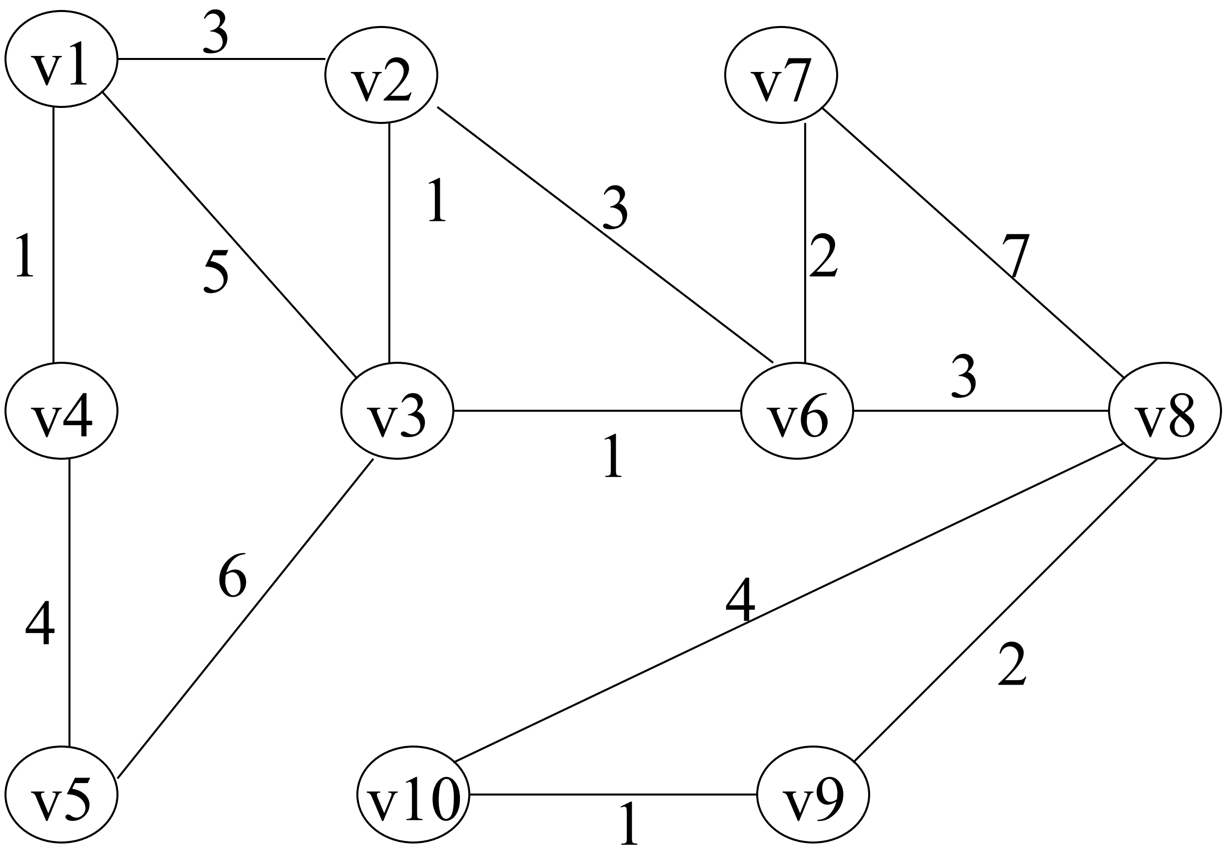
Single-source shortest-path problem:

Given as input a weighted graph, G = (V, E), and a distinguished starting vertex, s, find the shortest weighted path from s to every other vertex in G.

Dijkstra's algorithm (also called uniform cost search)

- Use a priority queue in general search/traversal
- Keep tentative distance for each vertex giving shortest path length using vertices visited so far.
- Record vertex visited before this vertex (to allow printing of path).
- At each step choose the vertex with smallest distance among the unvisited vertices (greedy algorithm).

Example Network



Dijkstra's Algorithm

- The pseudo code for Dijkstra's algorithm assumes the following structure for a Vertex object

```
class Vertex
{
  public List adj;       //Adjacency list
  public boolean known;
  public DisType dist;  //DistType is probably int
  public Vertex path;
  //Other fields and methods as needed
}
```

Dijkstra's Algorithm

```
void dijksra(Vertex start)
{
   for each Vertex v in V {
        v.dist = Integer.MAX_VALUE;
        v.known = false;
        v.path = null;
   }

   start.distance = 0;

   while there are unknown vertices {
        v = unknown vertex with smallest distance
        v.known = true;
        for each Vertex w adjacent to v
                if (!w.known)
                        if (v.dist + weight(v, w)< w.distance){
                                decrease(w.dist to v.dist+weight(v, w))
                                w.path = v;
                        }
   }
}
```

Correctness of Dijkstra's Algorithm

- The algorithm is correct because of a property of shortest paths:
- If $P_k = v_1, v_2, \ldots, v_j, v_k$, is a shortest path from v_1 to v_k, then $P_j = v_1, v_2, \ldots, v_j$, must be a shortest path from v_1 to v_j. Otherwise P_k would not be as short as possible since P_k extends P_j by just one edge (from v_j to v_k)
- P_j must be shorter than P_k (assuming that all edges have positive weights). So the algorithm must have found P_j on an earlier iteration than when it found P_k.
- i.e. Shortest paths can be found by extending earlier known shortest paths by single edges, which is what the algorithm does.

Running Time of Dijkstra's Algorithm

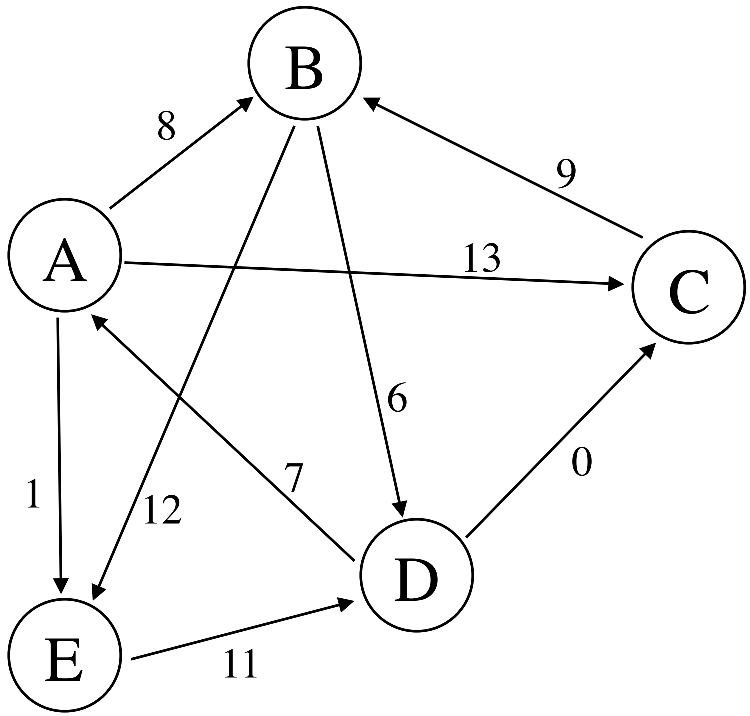
- The running time depends on how the vertices are manipulated.
- The main 'while' loop runs O(|V|) time (once per vertex)
- Finding the "unknown vertex with smallest distance" (inside the while loop) can be a simple linear scan of the vertices and so is also O(|V|). With this method the total running time is O ($|V|^2$). This is acceptable (and perhaps optimal) if the graph is dense (|E| = O ($|V|^2$)) since it runs in linear time on the number of edges.
- If the graph is sparse, (|E| = O (|V|)), we can use a priority queue to select the unknown vertex with smallest distance, using the deleteMin operation (O(lg |V|)). We must also decrease the path lengths of some unknown vertices, which is also O(lg|V|). The deleteMin operation is performed for every vertex, and the "decrease path length" is performed for every edge, so the running time is O(|E| lg|V| + |V|lg|V|) = O((|V|+|E|) lg|V|) = O(|E| lg|V|) if all vertices are reachable from the starting vertex

Dijkstra and Negative Edges

- Note in the previous discussion, we made the assumption that all edges have positive weight. If any edge has a negative weight, then Dijkstra's algorithm fails. Why is this so?
- Suppose a vertex, u, is marked as "known". This means that the shortest path from the starting vertex, s, to u has been found.
- However, it's possible that there is negatively weighted edge from an unknown vertex, v, back to u. In that case, taking the path from s to v to u is actually shorter than the path from s to u without going through v.
- Other algorithms exist that handle edges with negative weights for weighted shortest-path problem.

All-pairs shortest paths...

"Floyd-Warshall algorithm"



Matrix representation

TO

D^0

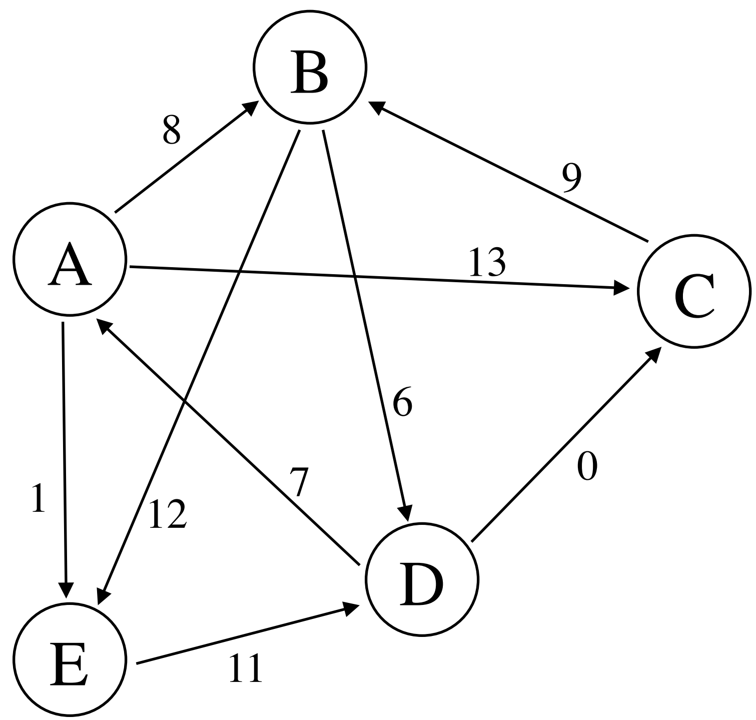
FROM		A	B	C	D	E
	A	0	8	13	-	1
	B	-	0	-	6	12
	C	-	9	0	-	-
	D	7	-	0	0	-
	E	-	-	-	11	0

All-pairs shortest paths...

$D^0 = (d_{ij}^0)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & 13 & - & 1 \\ - & 0 & - & 6 & 12 \\ - & 9 & 0 & - & - \\ 7 & - & 0 & 0 & - \\ - & - & - & 11 & 0 \end{pmatrix}$$

d_{ij}^k = shortest distance from i to j through $\{1, \ldots, k\}$



$D^1 = (d_{ij}^1)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & 13 & - & 1 \\ - & 0 & - & 6 & 12 \\ - & 9 & 0 & - & - \\ 7 & \boxed{15} & 0 & 0 & \boxed{8} \\ - & - & - & 11 & 0 \end{pmatrix}$$

All-pairs shortest paths...

$D^2 = (d_{ij}^2)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & 13 & \boxed{14} & 1 \\ - & 0 & - & 6 & 12 \\ - & 9 & 0 & \boxed{15 \quad 21} \\ 7 & 15 & 0 & 0 & 8 \\ - & - & - & 11 & 0 \end{pmatrix}$$

$D^4 = (d_{ij}^4)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & 13 & 14 & 1 \\ \boxed{13} & 0 & \boxed{6} & 6 & 12 \\ \boxed{22} & 9 & 0 & 15 & 21 \\ 7 & 9 & 0 & 0 & 8 \\ \boxed{18 \quad 20 \quad 11} & 11 & 0 \end{pmatrix}$$

$D^3 = (d_{ij}^3)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & 13 & 14 & 1 \\ - & 0 & - & 6 & 12 \\ - & 9 & 0 & 15 & 21 \\ 7 & \boxed{9} & 0 & 0 & 8 \\ - & - & - & 11 & 0 \end{pmatrix}$$

$D^5 = (d_{ij}^5)$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 8 & \boxed{12 \quad 12} & 1 \\ 13 & 0 & 6 & 6 & 12 \\ 22 & 9 & 0 & 15 & 21 \\ 7 & 9 & 0 & 0 & 8 \\ 18 & 20 & 11 & 11 & 0 \end{pmatrix}$$

to store the path, another matrix can track the <u>last intermediate vertex</u>

Floyd-Warshall Pseudocode

Input: $D^0 = (d_{ij}^0)$ (the initial edge-cost matrix)

Output: $D^n = (d_{ij}^n)$ (the final path-cost matrix)

```
for k = 1 to n          // intermediate vertices considered
  for i = 1 to n        // the "from" vertex
    for j = 1 to n      // the "to" vertex
```

$$d_{ij}^{k} = \min\{\ \underbrace{d_{ij}^{k-1}}_{\text{best, ignoring vertex k}},\ \underbrace{d_{ik}^{k-1} + d_{kj}^{k-1}}_{\text{best, including vertex k}}\}$$