

# 1 The Basic Counting Principles

- The Multiplication Rule

If an operation consists of  $k$  steps and  
 the first step can be performed in  $n_1$  ways,  
 the second step can be performed in  $n_2$  ways [regardless of how the first step was performed],  
 ...,  
 the  $k$ th step can be performed in  $n_k$  ways [regardless of how the preceding steps were performed],  
 then the entire operation can be performed in  $n_1 n_2 \cdots n_k$  ways.

**Example 1** *How many different results are there when spin 2 dice?*

**Example 2** *How many ways for this class to stand in line?*

- The Addition Rule

Suppose a finite set  $A$  equals the union of  $k$  distinct mutually disjoint subsets  $A_1, A_2, \dots, A_k$ .  
 Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

**Example 3** *How many ways to spin two dice to get an odd sum?*

- The Difference Rule

If  $A$  is a finite set and  $B$  is a subset of  $A$ , then

$$N(A - B) = N(A) - N(B).$$

**Example 4** *How many ways to spin two dice such that two dice are different?*

**Example 5** *If the chance for me to shoot a dart is  $\frac{1}{10}$ , assuming that I do it 10 times in a row, what is the chance for me to shoot the target?*

- Counting multiple in a controlled way.

**Example 6** *How many ways are there to select four people from your club to be the president, vice president, secretary and treasurer from 36 people? How many ways are there to select four people from your club with 36 people in total to form a committee?*

- Things are the same.

**Example 7** *These two problems are actually the same.*

- What is the maximum number of edges in a graph with  $n$  nodes?
- What is the number of pairs choosing from a group of  $n$  people?

## 2 Combinations and Permutations

- Permutation  $P(n, k)$ : choose  $k$  element from  $n$  and the order matters.  
For example  $P(36, 4) = 36 \times 35 \times 34 \times 33$ .

$$P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n - k)!}.$$

- Combination  $C(n, k)$ : choose  $k$  element from  $n$  and the order does NOT matter.

$$C(n, k) = \frac{n!}{(n - k)!k!}.$$

This is also denoted by  $\binom{n}{k}$ .

### 2.1 Permutation

**Example 8** *Permutations of the letters in a word:*

- How many ways can the letters in the word *COMPUTER* be arranged in a row?
- How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

**Example 9** *Permutations of selected letters of a word:*

- How many different ways can three of the letters of the word *BIKES* be chosen and written in a row?
- How many different ways can this be done if the first letter must be *B*?

**Example 10** *Prove that for all integers  $n \geq 2$ ,  $P(n, 2) + P(n, 1) = n^2$ .*

### 2.2 Combination

**Example 11** *How many unordered selections of two elements can be made from the set  $\{0, 1, 2, 3\}$ ? What are they?*

**Example 12** *Suppose two members of the group of twelve insist on working as a pair such that any team must contain either both or neither. How many five-person teams can be formed?*

**Example 13** *Suppose that two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?*

**Example 14** *What is the coefficient of  $x^3$  in  $(x + 1)^7$ ?*

## 2.3 Pascal's Triangle

$$\begin{array}{rcccccc}
 n = 0: & & & & & & 1 \\
 n = 1: & & & & & 1 & 1 \\
 n = 2: & & & & 1 & 2 & 1 \\
 n = 3: & & 1 & 3 & 3 & 1 \\
 n = 4: & 1 & 4 & 6 & 4 & 1
 \end{array}$$

### Theorem 1 (Binomial Theorem)

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

**Example 15** Prove that  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ .

**Example 16** Prove that  $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ .

**Example 17** Prove that  $\binom{2n}{2} = 2\binom{n}{2} + n^2$ .

**Example 18** Prove  $C(n, 0) + C(n, 1) + \cdots + C(n, n) = 2^n$  by a combinatorial argument.

**Exercise 1** Let  $C(n, k)$  be the number of ways to choose  $k$  objects from a set of  $n$ . **Prove by a combinatorial argument:**

1.  $C(n, n - k) = C(n, k)$
2.  $C^2(n, 0) + C^2(n, 1) + \cdots + C^2(n, n) = C(2n, n)$  (Hint: Use item 1)

## 3 Generalized Permutations and Combinations

### 3.1 Generalized Permutations

A generalized permutation allows to take an element more than once. Repetition is allowed.

**Example 19** Consider the following four problems. Are they related?

- How many different binary strings are there with length  $n$ ?
- What is the number of subsets of a set of  $n$  elements?
- How many different arrangements of an array with length  $n$ ?
- What is the number of ways to permute  $n$  values of two types with repetition?

What if we have more than 2 choices?

- What is the number of nodes in Tower of Hanoi graph with  $n$  disks?

- How many base 3 numbers of length  $n$ ?
- How many base 3 numbers of length  $n$  with at least 1 zero?
- How many base 10 numbers of length  $n$  with at least 2 zeros?
- How many base 10 numbers of length  $n$  with at least 3 zeros?

**Example 20** *Permutations of the letters in a word.*

- How many ways can the letters in the word CAB be arranged in a row?
- How many ways can the letters in the word ABRACADABRA be arranged in a row?
- How many ways to distribute 11 distinct items into 5 distinct boxes with 5 in  $B_1$ , 2 in  $B_2$ , 1 in  $B_3$ , 1 in  $B_4$ , and 2 in  $B_5$ ?

Given an alphabet contains  $k$  letters, the number of ways to write a word containing  $n_1$  times the first letter,  $n_2$  times the second,  $\dots$ ,  $n_k$  times the  $k$ th letter is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where  $n = n_1 + n_2 + \cdots + n_k$ .

### 3.2 Generalized Combinations

**Example 21** *Combination with repetition.*

- How many ways can I select 8 items from a basket containing large quantities of brownies, cookies, and muffins?

	brownies	cookies	muffins	
A	3	2	3	=8
B	0	1	7	=8
C	4	0	4	=8

- How many solutions are there for the equation  $x + y + z = 8$  where  $x, y, z \in \mathbb{N}$ ?
- How many ways are there to distribute  $n$  non-distinct things into  $m$  distinct categories?

The number of ways to fill  $r$  slots from  $n$  categories with repetition allowed is

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}.$$

In words,  $C(\#slots + \#transitions, \#slots) = C(\#slots + \#transitions, \#transitions)$ .

**Example 22** *Generalized combination.*

- How many ways can I fill a box holding 100 pieces of candy from 30 different types of candy?
- How many ways if I must have at least 1 piece of each type?

## 4 The Inclusion/Exclusion Rule

**Example 23** *Consider equation  $x + y + z = 8$  where  $x, y, z \in \mathbb{N}$  in Example 21.*

- How many solutions are there?
- How many solutions are there when  $x \geq 3$  and  $z \geq 1$ ?
- How many solutions are there when  $x < 3$ ?

The Inclusion/Exclusion Rule for Two or Three Sets

If  $A, B,$  and  $C$  are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

**Example 24** *Recall the problems in Example 19.*

- How many base 10 numbers of length  $n$  without zeros?
- How many base 10 numbers with at least a zero and at least a one?
- How many permutation are there of  $aaabbbccc$  without 3 consecutive same letter?
- How many solutions are there for the equation  $x + y + z = 8$  where  $x, y, z \in \mathbb{N}$ ,  $x < 3$ ,  $y < 4$ , and  $z < 5$ ?

**Example 25 (Derangement)** *A derangement of the set  $\{1, 2, \dots, n\}$  is a permutation that moves every element of the set away from its “natural” position. What is the number of derangement of a set with size  $n$ ?*

## 5 The Pigeonhole Principle

### Example 26

- Show that 2 people in this room were born on the same day of the week.
- The test score is from 0 to 100. To guarantee 3 people get the same score, how many people take the test?
- There are 51 houses on the street. Their addresses are between 1000 and 1099 inclusive. Show that there are at least two houses with consecutive addresses.
- Let  $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ .
  1. If five integers are selected from  $A$ , must at least one pair of the integers have a sum of 9?
  2. If four integers are selected from  $A$ , must at least one pair of the integers have a sum of 9?
- There are 42 students who are to share 12 computers. Each student uses exactly 1 computer, and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.
- There are  $n$  computers on a network. Show that at least 2 computers are connected to the same number of other computers.
- A computer goes down at least 1 time every day. During a month (30 days), it goes down at most 45 times. Show that it goes exactly 14 times in some consecutive period of days.

## 6 Probability

**Definition 1 (Sample Space and Event)** A sample space is the set of all possible outcomes of an experiment. An event is a subset of the sample space. For an event  $A$ , the probability of outcomes of event  $A$  is

$$P(A) = \frac{\text{total number of outcomes that constitute the event } A}{\text{total number of possible outcomes}}$$

### Probability Axioms

Let  $S$  be a sample space, A **probability function**  $P$  from the set of all events in  $S$  to the set of real numbers satisfies the following three axioms: For all events  $A$  and  $B$  in  $S$ ,

1.  $0 \leq P(A) \leq 1$
2.  $P(\emptyset) = 0$  and  $P(S) = 1$
3. If  $A$  and  $B$  are disjoint ( $A \cap B = \emptyset$ ), then the probability of the union of  $A$  and  $B$  is  $P(A \cup B) = P(A) + P(B)$ .

**Example 27**

- When you throw a die, you get 1 out of 6 different values. What is the chance for you to get a 1 or 3?
- Consider tossing a fair coin, i.e., the chance of getting a head is 50%. What is the chance to get a head 5 times in a row?
- If you are a 60% free throw shooter. What is the probability that you make two free throws in a row? What is the probability that you miss both? What is the probability that you make exactly one shot?

**Definition 2** Two events are independent when the outcome of one does not affect the outcome of the other.

**The Multiplication Principle**

The probability of two independent events **both** happening is the product of their individual probabilities.

**Example 28** Suppose that you roll a die 6 times. What is the probability that you roll at least one six?

**Probability of the Complement of an Event**

If  $A$  is any event in a sample space  $S$ , then

$$P(A^c) = 1 - P(A).$$

**Probability of a General Union of Two Events**

If  $S$  is any sample space and  $A$  and  $B$  are any events in  $S$ , then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

**Definition 3** Suppose the possible outcomes of an experiment, or random process, are real numbers  $a_1, a_2, a_3, \dots, a_n$ , which occur with probabilities  $p_1, p_2, p_3, \dots, p_n$ . The **expected value** of the process is

$$\sum_{k=1}^n a_k p_k = a_1 p_1 + a_2 p_2 + a_3 p_3 + \dots + a_n p_n.$$

**Example 29** You are watching the NBA finals with your friends. Kobe Bryant is on the free throw line with no time left on the clock. Suppose that his free throw shooting percentage is an abysmal 58% (only for the purposes of this exercise). He needs to make both shots to win the game. Your friend bets you \$20 they don't win the game, and he promises you 2:1 odds. Do you take the bet?

Indeed, Kobe makes both free throws with probability  $0.58 \times 0.58$ . He makes the first and misses the second with probability  $0.58 \times 0.42$ . He misses the first and hits the second with probability  $0.42 \times 0.58$ . And he misses both with probability  $0.42 \times 0.42$ . This can be viewed algebraically as:

$$1.0 \times 1.0 = (0.58 + 0.42) \times (0.58 + 0.42) = (0.58 \times 0.58) + (0.58 \times 0.42) + (0.42 \times 0.58) + (0.42 \times 0.42).$$

**Example 30 (Craps)** *The game of craps is a popular casino dice game which is played with two dice. Here are the rules to determine whether you win or not.*

- *On the first roll, the player wins if the sum of the dice is 7 or 11. These two numbers are called naturals.*
- *The player loses if the sum is 2, 3, or 12, called craps.*
- *Any other roll is called the “point” and the game continues. On each subsequent roll, the player wins if he or she rolls the point again. The player loses by rolling 7. Any other roll is ignored and the game continues.*

*You rolled: 8  
Your point is 8  
Your rolled: 3  
Your rolled: 10  
Your rolled: 8  
You win!*

*You rolled: 6  
Your point is 6  
Your rolled: 5  
Your rolled: 12  
Your rolled: 3  
Your rolled: 7  
You lose!*

1. *What is the probability you win a game of craps on the first roll?*
2. *What is the probability you win a game of craps on the second roll?*
3. *What is the probability you win a game of craps after the third roll?*
4. *Generalize the expression for an infinite number of rolls.*

*Is there an easier way to calculate these probabilities - one does not need an infinite series? Yes, conditional probability.*

In the previous example, if we care about only outcomes that are 7 and 4, then there are effectively no longer 36 outcomes when we roll, but instead 9: six outcomes for 7 and three for 4. So we should expect to roll a 4 before a 7 three out of nine times, or  $1/3$ . We say that the probability of rolling a 4, *on the condition* that your roll is either 7 or 4, is  $1/3$ . Therefore, the probability of winning by starting with 4 is  $\frac{1}{12} \times \frac{1}{3} = \frac{1}{36}$ .

Next we consider situations when you are given more information and how probabilities change.

**Example 31** *Consider a couple with two children.*

- *What is the probability that both children are boys?*
- *If you know that one child is a boy, what is the probability that both children are boys?*
- *If the older child is a boy, what is the probability that both children are boys?*



**Definition 4** Let  $A$  and  $B$  be events in a sample space  $S$ . If  $P(A) \neq 0$ , then the **conditional probability of  $B$  given  $A$** , denoted  $P(B|A)$ , is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

**Example 32** Recalculate the probability of winning craps using the conditional probability.

**Theorem 2 (Bayes' Theorem)** Suppose that a sample space  $S$  is a union of mutually disjoint events  $B_1, B_2, B_3, \dots, B_n$ , suppose  $A$  is an event in  $S$ , and suppose  $A$  and all the  $B_k$  have nonzero probabilities. If  $k$  is an integer with  $1 \leq k \leq n$ , then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + \dots + P(A|B_n)P(B_n)}.$$

Let's see how we can derive Bayes' Theorem. Note that that conjunction is commutative. That is,

$$P(A \cap B) = P(B \cap A)$$

for any events  $A$  and  $B$ . Next, we write the probability of a conjunction:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Since  $P(A \cap B) = P(B \cap A)$ , we have

$$P(B|A)P(A) = P(A \cap B) = P(B \cap A) = P(A|B)P(B).$$

Therefore,  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ .

Most medical tests occasionally produce incorrect results, called **false positives** and **false negatives**. When a test is designed to determine whether a patient has a certain disease, a false positive result indicates that a patient has the disease when the patient does not have it. A false negative result indicates that a patient does not have the disease when the patient does have it.

**Example 33** Consider a medical test that screens for a disease found in 5 people in 1,000. Suppose that the false positive rate is 3% and the false negative rate is 1%. Then 99% of the time a person who has the condition tests positive for it, and 97% of the time a person who does not have the condition tests negative for it.

1. What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?
2. What is the probability that a randomly chosen person who tests negative for the disease does not indeed have the disease?